Vol. III 1932

THE ANNALS
of
MATHEMATICAL STATISTICS

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DISTRIBUTION OF THE MEANS DIVIDED BY THE STANDARD DEVIATIONS OF SAMPLES FROM NON-HOMOGENEOUS POPULATIONS

By

G. A. BAKER

In a previous paper[1] the distributions of the means and variances, means squared and variances of samples of two drawn from a non-homogeneous population composed of two normal populations have been discussed. It is the purpose of this paper to discuss similarly the distribution of the means of samples of two measured from the mean of the population divided by the standard deviations of the samples for such parent populations and to present experimental results for samples of four.

CASE $n = 2$

Suppose that a population represented by

(1)

$$f(x) = \frac{N_1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x+m_1)^2}{\sigma_1^2}} + \frac{N_2}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x-m_2)^2}{\sigma_2^2}}$$

$$-N_1 m_1 + N_2 m_2 = 0$$

is considered. If $n-s$ individuals come from the first component and s from the second in drawing samples of n and if $\bar{m}$ is the mean of the sample measured from the mean of the population and $\bar{\sigma}$ is the standard deviation of the sample,[2] then

(2)

$$\frac{\bar{m}}{\bar{\sigma}} = \frac{-(n-s)\bar{m}_1 + s\bar{m}_2}{\sqrt{n}\left[(n-s)\bar{\sigma}_1^2 + s\bar{\sigma}_2^2 + \frac{(n-s)s}{n}(\bar{m}_1 + \bar{m}_2)^2\right]^{\frac{1}{2}}}$$

---

[1] *Annals of Mathematical Statistics*, Vol. 2, No. 3, Aug. 1931.

[2] "Random Sampling from Non-Homogeneous Populations"—*Metron*, Vol. 8, No. 3, p. 6.

where $\bar{m}_1, \bar{m}_2, \bar{\sigma}_1^2$ and $\bar{\sigma}_2^2$ are estimates of the corresponding parameters of (1).

For the case $n = 2$ when both individuals come from the first component of (1) it is known that the distribution of the means divided by the standard deviations of the samples is proportional to

(3)

$$\frac{du}{1+u^2}$$

the origin of u being taken at $-\frac{m_1}{\sigma_1}$. Similarly, when both individuals of the sample come from the second component, $\frac{\bar{m}}{\bar{\sigma}}$ is distributed as proportional to

(4)

$$\frac{dw}{1+w^2}$$

the origin of w being taken at $\frac{m_2}{\sigma_2}$.

When one individual comes from each component (2) becomes

(5)

$$\frac{\bar{m}}{\bar{\sigma}} = \frac{-\bar{m}_1 + \bar{m}_2}{\sqrt{2}\left[\frac{1}{2}(\bar{m}_1+\bar{m}_2)^2\right]^{\frac{1}{2}}}$$

because no estimate of the standard deviations of the components of (1) can be made from one individual. The distribution of $\bar{m}_1$ is proportional to the first component of (1), and $\bar{m}_2$ is distributed as proportional to the second component. The distributions of $\bar{m}_1$ and $\bar{m}_2$ are independent.

Expression (5) can be rewritten as

(6)

$$\frac{\bar{m}}{\bar{\sigma}} = 1 - \frac{2}{1+\frac{\bar{m}_2}{\bar{m}_1}}.$$

Put

$$\frac{\bar{m}_2}{\bar{m}_1} = v.$$

(7)

$$\frac{\bar{m}_2}{\bar{m}_1} = v.$$

If the distribution of v is found, the distribution of $z = \frac{\bar{m}}{\bar{\sigma}}$ may be found by making the transformation

(8)

$$z = 1 - \frac{2}{1+v}$$

or

(9)

$$v = -1 - \frac{2}{z-1}$$

and

(10)

$$dv = \frac{2}{(z-1)^2}\, dz.$$

The distribution of v is a special case of the distribution of an index both of whose components follow the normal law. That is, we seek the distribution of

$$v = \frac{y}{x}$$

x and y being distributed as

(11)

$$z_0 e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(x-\bar{x})^2}{\sigma_1^2} - 2r\frac{(x-\bar{x})}{\sigma_1}\frac{(y-\bar{y})}{\sigma_2} + \frac{(y-\bar{y})^2}{\sigma_2^2}\right]}.$$

This distribution may be obtained as follows.

Lemma I.[2,3] If two variables x and y, $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$

[2] (Loc. cit.)

[3] Baten, W. D. "Combining Constant Probability Functions"—*American Mathematical Monthly*, Oct., 1930.

are so related that the probability of an x in dx and of a y in dy is

$$f(x,y)\,dx\,dy$$

then the probability that $v = x - y$ is in dv is proportional to

$$\left[\int_{-\infty}^{\infty} f(v+y, y)\,dy\right]dv.$$

Consider, first, the portion of (11) in the first quadrant. Put

$$v = \frac{y}{x}$$

and take the logarithm of each side, thus

$$\log v = \log y - \log x. \tag{12}$$

Put

$$I = \log v$$
$$w = \log y$$
$$u = \log x$$

and (12) becomes

$$I = w - u \tag{13}$$

where the range of w and u is $-\infty$ to $+\infty$. The equation of the correlation surface of w and u is proportional to

(14)

$$F(w,w) = e^{u}e^{w}e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(e^u-\bar{x})^2}{\sigma_1^2} - 2r\frac{(e^u-\bar{x})}{\sigma_1}\frac{(e^w-\bar{y})}{\sigma_2} + \frac{(e^w-\bar{y})^2}{\sigma_2^2}\right]}.$$

Hence, $F(u, I+u)du$ when the transformation $e^u = x$ is made, becomes

(15)

$$xe^{I}e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(x-\bar{x})^2}{\sigma_1^2} - \frac{2r(x-\bar{x})}{\sigma_1}\frac{(xe^I-\bar{y})}{\sigma_2} + \frac{(xe^I-\bar{y})^2}{\sigma_2^2}\right]}dx$$

where x ranges from 0 to ∞. By the application of Lemma I the proportional probability of a value of v in dv when x and y are both positive is obtained by integrating (15) from 0 to ∞ with respect to x and making the transformation $v = e^I$. Thus,

(16)

$$\frac{\sigma_1\sigma_2\sqrt{1-r^2}}{d}\, e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{\bar{x}^2}{\sigma_1^2}-2r\frac{\bar{x}\bar{y}}{\sigma_1\sigma_2}+\frac{\bar{y}^2}{\sigma_2^2}\right]}$$

$$+\frac{b}{a^{3/2}}e^{-\frac{(\bar{x}v-\bar{y})^2}{2a}}\int_0^{\frac{b}{\sigma_1\sigma_2\sqrt{1-r^2}\sqrt{a}}} e^{-\frac{1}{2}z^2}\,dz$$

$$+\frac{\sqrt{\pi}}{\sqrt{2}}\frac{b}{a^{3/2}}e^{-\frac{(\bar{x}v-\bar{y})^2}{2a}}$$

where

$$a=\sigma_2^2-2r\sigma_1\sigma_2 v+\sigma_1^2 v^2$$

$$b=(\sigma_1^2\bar{y}-r\sigma_1\sigma_2\bar{x})v+(\sigma_2^2\bar{x}-r\sigma_1\sigma_2\bar{y})$$

is obtained. The distribution of v if both x and y are negative (and hence v positive) is the same as (16) except that the last term is reversed in sign. Thus, for v positive the distribution of v is proportional to two times the first two terms of (16). If v is negative, i.e. x negative and y positive or x positive and y negative, (16) is obtained in one case and (16) with the sign of the last term changed in the other case. That is, the distribution of v is proportional to the first two terms of (16) when v ranges from $-\infty$ to $+\infty$.

In our case $r=0$, $m_1=\bar{x}$, $m_2=\bar{y}$. Hence, the distribution of v becomes proportional to

(17)

$$\frac{\sigma_1\sigma_2\, e^{-\frac{1}{2}\left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}\right]}}{(\sigma_2^2+\sigma_1^2v^2)}$$

$$+\frac{\sigma_1^2 m_2 v+\sigma_2^2 m_1}{(\sigma_2^2+\sigma_1^2v^2)^{3/2}}\, e^{-\frac{(m_1v-m_2)^2}{2(\sigma_2^2+\sigma_1^2v^2)}}\int_0^{\frac{\sigma_1^2 m_2 v+\sigma_2^2 m_1}{\sigma_1\sigma_2\sqrt{\sigma_2^2+\sigma_1^2v^2}}} e^{-\frac{1}{2}z^2}\,dz.$$

From (8), (9), (10), and (17) z is distributed as proportional to

(18)

$$\frac{\sigma_1\sigma_2}{A}\, e^{-\frac{1}{2}\left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}\right]} + \frac{B}{A^{3/2}}\, e^{\frac{-\left[z(-m_1-m_2)-(m_1-m_2)\right]^2}{2A}} \int_0^{\frac{B}{\sigma_1\sigma_2 A^{1/2}}} e^{-\frac{1}{2}u^2}\, du$$

where

$$A = (\sigma_1^2)+\sigma_2^2)z^2+2(\sigma_1^2-\sigma_2^2)z+(\sigma_1^2+\sigma_2^2)$$

and

$$B = z(\sigma_2^2 m_1 - \sigma_1^2 m_2) - (\sigma_1^2 m_2 + \sigma_2^2 m_1).$$

The origin for z is at the mean of population (1).

Thus the distribution of $\frac{m}{\sigma}$ for samples of two drawn from a population represented by (1) has been completely determined as being proportional to

(19)

$$k_1(3) + k_2(4) + k_3(18).$$

Let A_1, A_2 and A_3 be the respective areas under the curves represented by the three terms of (19). Then k_1, k_2, and k_3 are to be so determined that

$$A_1 + A_2 + A_3 = N$$

where N is the total number of samples considered, and that

$$\frac{1}{A_1} = \frac{k^2}{A_2} = \frac{2k}{A_3}$$

where

$$k = \frac{N_2}{N_1}.$$

Expression (19) indicates, in general, that if the means of the components do not coincide and if one component is not large compared with the other, both tails of the distribution of the means of samples measured from the mean of the population divided by the standard deviations of the samples are heavier for populations of the type (1) than for normal populations. In case the means of the components coincide one tail will be heavier $(\sigma_1^2 \neq \sigma_2^2)$. In any case at least one tail will be heavier.

EXPERIMENTAL RESULTS

Samples of four were drawn from a population approximately represented by

(1)

$$f(x) = \frac{648}{5\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-15.5)^2}{25}} + \frac{648}{5\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-32.5)^2}{25}}$$

which is the same as Population I in the first reference. These samples were drawn by throwing dice. The means of these samples were calculated and referred to the mean of (1) as an origin. The standard deviations of the samples were obtained and $\frac{\bar{m}}{\sigma}$ calculated. A grouped frequency distribution of 1038 of these values is presented in Chart I and in Table I. Large values are obtained more frequently than would be expected from a normal population.

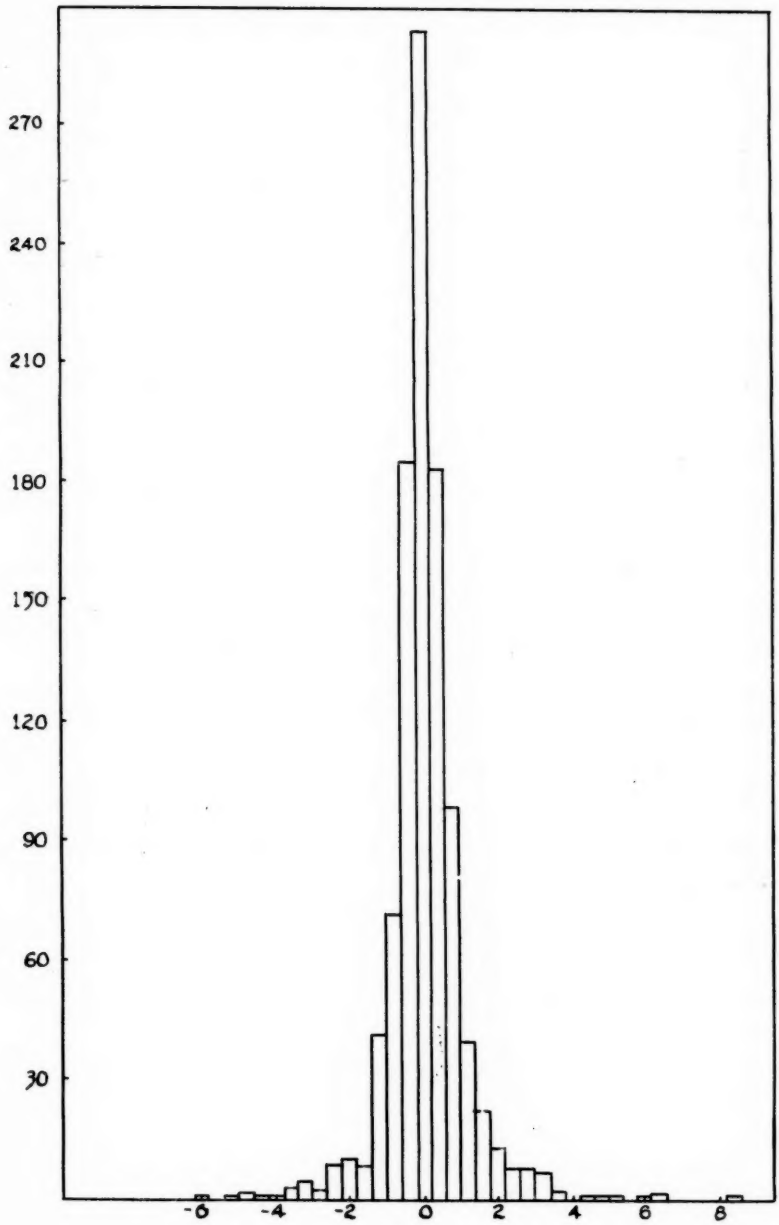
TABLE I

Grouped Frequency distribution of 1038 Values of $\overline{m}/\overline{\sigma}$ for Samples of Four from Population I

Middle of Interval	Frequency
—6.0	1
—5.6	0
—5.2	1
—4.8	2
—4.4	1
—4.0	1
—3.6	4
—3.2	6
—2.8	4
—2.4	10
—2.0	11
—1.6	9
—1.2	42
—0.8	71
—0.4	185
0	294
0.4	183
0.8	100
1.2	40
1.6	23
2.0	14
2.4	9
2.8	9
3.2	8
3.6	4
4.0	0
4.4	1
4.8	1
5.2	1
5.6	0
6.0	1
6.4	2
6.8	0
7.2	0
7.6	0
8.0	0
8.4	1

CHART I

Grouped Frequency Distribution of 1038 Vaiues of $\overline{m}/\overline{\sigma}$ for Samples of Four from Population I



A STATISTICAL APPROACH TO MATHEMATICAL FORMULATION OF DEMAND-SUPPLY-PRICE RELATIONSHIPS

By
ROBERT W. BURGESS

A scientific approach to the practical problem of forecasting the prices of commodities clearly requires the development of methods of a somewhat mathematical type for analyzing the relationships between demand, supply, cost, and price. In the case of cotton and other annual crop agricultural commodities, the multiple correlation, link-relative and trend-ratio methods as applied by Moore, Schultz, B. B. Smith, Ezekiel, Holbrook and E. J. Working, and others, have demonstrated their worth. But for copper, lead, rubber, and similar commodities not on an annual crop basis, where quantity produced, or the quantity available, in a given period cannot logically be considered as the supply linked to the average price of that period, the method seems inapplicable and another type of approach is necessary. For this reason, and because of the failure of price to function as expected as a major regulator in our present money economy, it seems worth while to attempt to develop a general mathematical procedure involving cost, demand and supply functions and to analyze elasticity of supply and elasticity of demand as a mathematician naturally does. But some mathematical studies along these lines have not seemed to represent a truly scientific approach to the problem, however helpful they may be in suggesting potentially valuable ideas.

The mathematical economist of the non-statistical type sometimes seems to believe that he has contributed to the solution of economic problems if he finds an answer in the form of a mathematical equation with undetermined constants. The determination of these constants is left as a sec-

ondary step to the statistician. But in certain cases, at least, the determination of these constants would be at least as difficult as the original problem. Moreover, the assumptions made in setting up the equations, while not always stated explicitly, have not been shown to constitute a sufficiently close aproximation to actual non-exceptional conditions for the analysis to be useful. It may be suggested that, according to correct scientific procedure, careful statistical analysis of known and knowable conditions is necessary *before* worth while mathematical formulation can be attempted.

To bring out my point of view by discussion of a particular equation, Professor Evans, in his "Introduction to Mathematical Economics," uses as a cost function, that is to say, a function stating total cost for goods produced in a given unit of time, in terms of the quantity produced, u,

$$q(u) = Au^2 + Bu + C$$

This formula involves several unstated assumptions:

(a) That the same continuous formula will apply over the entire range of u which may appropriately be considered in discussing a particular problem. Actually, several points of discontinuity would be more normal. In fact the natural method of statement of this relation might be merely a number of discrete points rather than a function defined for all values of u. For instance, under the conditions of a particular problem, increase in production might be accomplished solely by increasing the number of machines engaged in that process. If so, any rate of output other than an integral multiple of the normal output of the machine might represent inefficient operation and therefore be barred from further discussion. More generally, I should expect a very high unit cost for low values of u, when pro-

duction is on a job basis, with abrupt downward steps as quantity production methods are applied, and finally a nearly horizontal line.

(b) That a term of the type Au^2 is worth considering in the typical case. Some analysis seems called for as to when this term is appropriate. In mining and agriculture, some such increase of cost with increasing quantity undoubtedly does occur, but it is difficult to imagine practical cases of factory production in which unit cost increases with quantity at any such rate if advance notice of contemplated increase has been given. If production beyond present normal plant capacity is desired, such forced production might involve some temporary increase in unit cost as production increased, but in these days such a case would be unusual.

(c) That the variation of unit cost with quantity is important enough to justify singling that out as the single factor of variation, although actually my impression is that for rather broad ranges of quantity, other factors are more significant causes of variation in costs. Such factors include regularity of production rate, the weather, labor conditions, character of supervision, and management pressure reflecting price conditions for the product.

(d) That $\frac{C}{u}$ is an adequate expression for the element of unit cost which decreases as u increases. Accounting discussions of total cost usually emphasize, in addition to constant overhead, certain elements of cost which increase somewhat with the quantity produced, but not in direct proportion. There is also the very sharp reduction in unit costs when production is first initiated. As a first hypothesis as a basis for detailed statistical study, I suggest that a term of the type Ce^{-ku} might be useful, either alone

or with the term $\frac{C}{u}$, to cover the elements of unit costs which decrease as u increases.

(e) That some definition of cost of a logical nature can be framed and applied to this case.

The statistical aproach to a cost function might well show that the range of applicability of a continuous formula is rather narrowly limited and that a careful statement as to attendant conditions is at least as important for reliability of the results as precise determination of the constants. Unfortunately, the results of statistical experiments along these lines are not available. In these days of stiff competition and rigorous government regulation, industrial concerns will be reluctant to permit publication of the kind of analysis of costs which is essential to what seems to be the correct scientific approach to this problem. It is hoped, however, that the preceding discussion has been specific enough to show the nature of the analysis which I think should precede the formulation of an expression of functional relationship, and to suggest the kind of discussion which a mathematical economist should include in his results.

Returning to the general criticism that the treatment of supply-demand-price relationships by certain mathematical economists is not truly scientific, let us review in broad lines the history of scientific progress in those lines where it has admittedly been successful. The steps have been about as follows:

1. Creation of a serviceable mechanism for the measurement of data. In the case of astronomy, for instance, this required the invention of the telescope and general agreement on angular measurement.

2. Careful making and recording of observations with this approved system of measurement, the observations

being in certain cases of events over which the scientist has no control and in other cases of experiments whose conditions could be modified at his convenience.

3. Derivation of empirical laws from these data.

4. Discovery of fundamental principles.

It is true that these steps are not in fact separated as completely as the outline might suggest, and that attempts at the discovery of fundamental principles often help in formulating the plan according to which observations are recorded and give the workers a motive for intensive effort. I think the statement will stand, however, that very few discoveries of fundamental principles have been made until substantial results have been secured under the (1), (2) and (3) headings.

For example, Newton's astronomical laws were discovered after Huyghens and others had made the telescope a useful instrument, Tycho Brahe had made an enormous number of observations, and Kepler had deduced empirical laws for those observations.

Again, in actuarial science, the whole structure of modern life insurance became possible only after careful vital statistics had been recorded for many years and analyzed by the empirical laws of Gompertz and Makeham.

On the other hand, when we turn to treatises on theoretical economics or to books and articles on mathematical economics, there seems to be no trace of the careful recording of observations or their analysis by empirical laws as the basis for their theoretical discussions. I admit, of course, that mathematical formulas are stated which look like empirical laws, but no references are given to any studies justifying these particular formulations. If we imagine ourselves starting the scientific procedure described above as the basis for arriving at real economic laws, we note almost at once

that agreement as to the meaning of fundamental terms has not yet been secured. For instance, cost of production is one idea which is fundamental in analysis of demand-supply-price relationships, but cost accountants and economists are by no means agreed among themselves as to what the term should cover. Under the circumstances, it seems to me that the most profitable scientific approach *at present* would be to analyze various relationships which can be put on a quantitative basis, with special attention to noting all the special circumstances of the cases analyzed. For example:

(a) In many cases it would probably be possible to study the relationship between the price of a manufactured article and the price of the raw material or the raw materials used in making it. A simple case which I have actually done in my office and used in price estimating is determining the price of cotton yarn in terms of that of raw cotton. We find it advantageous to compute the cotton yarn price according to the formula, compare that with the actual price and note the relation to general business conditions or to competitive conditions within the industry. The statistical methods required for a problem of this type are obvious, but relatively little, I believe, has been done on this line. Broad comparisons of index numbers of prices of finished goods and raw materials I regard as another kettle of fish altogether.

(b) Relationship of change in price to change in stocks. Any study of actual data of the commodities, such as copper, lead, and rubber, shows that price tends to decline when stocks increase and rise when stocks decrease. A first step in the quantitative approach to price forecasting is to obtain a more precise formulation of this correspondence. It may be noted that some rather vague mathemat-

ical ideas come to the surface in discussions of these relationships. For instance, if in a given month production has decreased and consumption has increased, it is sometimes said that these are two arguments for higher prices, and prices are expected to rise. But it may happen in certain cases that even with such a decrease in production and increase in consumption the month's production still exceeds consumption and stocks are increasing. On the whole, then, the monthly figures point to lower rather than higher prices, and it is a useful duty of the mathematician to point this out.

An audience of mathematicians probably regards the preceding illustration as trifling. I bring it up to illustrate the fact that progress toward a more mathematical attitude in commodity forecasting must proceed step by step. A more advanced stage in the quantitative formulation is, of course, to determine the equation connecting change in price and change in stocks as reported monthly. This also has actually proved useful.

(c) Exact definition of the phrase "cost of production" as actually effective when the problem is:

1. Establishing an appropriate price under regulated monopoly conditions.

2. Determining which manufacturing or mining enterprise will survive.

3. Shutting down established sources of supply.

4. Creating new sources of supply.

As I see it, there are at least four costs of production, eacl of which is important under certain circumstances.

1. Complete economic cost, including interest on er tire investment at an appropriate rate.

2. Economic cost, excluding interest on the capital value of ownership.

3. Out-of-pocket expense, which excludes depreciation charges in excess of actual replacements in the period considered, interest on investment, and design or development expenses.

4. Economic cost plus reward for the enterpriser over and above interest at a reasonable rate on his investment.

Roughly speaking, it may be that these are in order closely related to the costs required for the problems stated above.

(d) Analysis of changes in cost at different price levels. One point which has been emphasized by the experience of the past year or two is the fact that costs are by no means kept constant when the price varies. For instance, the best information available two years ago suggested that at 18¢ New York, certain producers of rubber would begin to drop out because of costs above that figure. As a matter of fact, they seem to have been able to change their costs. Such changes are possible by several means, for instance:

1. Wage differentials varying with market price.

2. Bonuses for officials varying with profits.

3. Exploitation of best ores or plantations in times of lower prices.

4. Increased pressure for efficiency when essential.

(e) Careful analysis of the relations of cost to quantity produced with the consideration of:

1. The time ahead that the quantity is known to be required.

2. Continuous production versus intermittent production, perhaps in lots whose size has been determined to

be most economical in view of manufacturing and distributing conditions.

3. Mass production or production in smaller quantities by trained mechanics with attention to the discontinuity in costs per unit when the transition from one type of production to another occurs.

In view of the considerations suggested in (1), (2), and (3), it does not seem to me that the assumption of an algebraic formula connecting cost with quantity produced represents an adequate realistic formulation of the problem.

(f) Analysis of way elasticity of supply or demand actually works. Such an analysis seems to me essential before a definition of coefficient of elasticity is framed or made the basis for elaborate developments. Recent experience has, I think, shown:

1. Elasticity of demand does not mean shrinkage of demand with high prices to anything like the extent expected.

2. Elasticity of supply functions much more slowly than expected at the low-price end of the spectrum. The frictional factors include sympathy with employees who would lose their jobs if economically unprofitable production ceased, reserve funds which permit companies to continue operations when even out-of-pocket expenses are not being fully recovered, bank willingness to lend on commodity stocks which are not, in fact, marketable within a short time, and the cost of shutting down and reassembling a working force.

Most of the matters discussed in (a) to (f) above cannot be analyzed fully on the basis of published records. Moreover, in view of the fact that price information is part of the life blood of any particular business, it will prove difficult for outsiders to secure much of the information which would

be needed to complete the analysis. A possible plan would be to place research specialists in industrial or marketing concerns to study actual data. It is not probable that the best research of this type can be done in academic halls as a sideline to teaching. It is also unlikely that corporation officers with an ax to grind can themselves complete the scientific analysis of such material. It seems clear, however, that a satisfactory solution requires a coordination of both points of view.

Summarizing the point of view I have tried to outline, I believe that the analysis of cost-price-supply-demand relationships should be relatively more inductive than it has been, especially in the type of theoretical work classified as mathematical economics. On the other hand, I think that the deductive approach is worth while, that we should try to formulate general principles in this field, and that the ultimate ideal involves a mathematical form,—though, perhaps, when it really covers price situations as they are, a mathematical form somewhat different from those required in the physical, chemical and astronomical sciences.

Robert W. Burgess

THE DISTRIBUTIONS OF THE PRECISION CONSTANT AND ITS SQUARE IN SAMPLES OF n FROM A NORMAL POPULATION

By

H. M. FELDMAN
Washington University

INTRODUCTION

The following paper is a study of the properties of the distributions of the precision constant and its square in samples of from a normal population. The properties studied are (1) modes and optimum values, (2) the first four moments, (3) skewness and flatness, and (4) medians and quartiles.

The distribution curves shown in the figure are for $n=4$, 10, and 25. All the curves are drawn together and to the same scale, so that a graphical comparison of the two distributions can be easily made for both the same and different values of n. The numerical values for the various parameters given in the tables are for $n=4$, 10, 25, and 100, except in the case of the medians and quartiles where the values for $n=100$ are omitted, and in case of $n=4$, no moments higher than the second exist for the precision constant, and none higher than the first for the precision constant squared distribution.

1. *Distributions*

Let us denote the standard deviation, precision constant, and the precision constant squared of the parent population by σ, h, and u, respectively, and those of a sample from the given population by s, h, and u, respectively. The standard deviation, s, is then defined in terms of the variates, $x_1, x_2, \cdots\cdots, x_n$ and its mean $\bar{x}$ by the equation

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \quad \cdots\cdots\cdots\cdots\cdots \quad (1.0)$$

We also have the following well known relations between S, H, and U, or s, h, and u:

$$s = \frac{1}{\sqrt{2}h} \quad \ldots\ldots\ldots\ldots \quad (1.1)$$

$$s^2 = \frac{1}{2u}, \text{ or } u = h^2 \quad \ldots\ldots\ldots \quad (1.2)$$

The distribution of s as given by R. A. Fisher* and others is

$$f(s)ds = \frac{n^{\frac{n-1}{2}} s^{n-2} e^{-\frac{ns^2}{2s^2}} ds}{2^{\frac{n-3}{2}} S^{n-1} \Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots \quad (1.3)$$

where n is the number of items in the sample.

The distribution of s and the transformations (1.1) and (1.2) enable us to find the distributions of the precision constant h, and its square u. Thus, using (1.1) and (1.3) we get

$$F(h)dh = \frac{n^{\frac{n-1}{2}} H^{n-1} h^{-n} e^{-\frac{nH^2}{2h^2}} dh}{2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots \quad (1.4)$$

and by means of (1.2) and (1.3) we find

$$\phi(u)du = \frac{n^{\frac{n-1}{2}} U^{\frac{n-1}{2}} u^{-\frac{(n+1)}{2}} e^{-\frac{nU}{2u}} du}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots \quad (1.5)$$

2. *Modal and Optimum Values*

We shall now obtain some of the properties of the distributions of our parameters. The simplest of the properties of any

*See, for example, R. A. Fisher, Applications of "Student's" distribution, Metron, Vol. 5, No. 3, (Dec. 1, 1925), pp. 90-104.

continuous distribution is its modal value or the abscissa of the maximum ordinate of the curve. Denoting the modal values of the distributions (1.4) and (1.5) by $\tilde{h}$ and $\tilde{u}$ respectively, we find from the condition for an extremum

$$\frac{dF}{dh}=0, \quad \frac{d\phi}{du}=0,$$

$$H=\tilde{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.0)$$

$$U=\frac{n+1}{n}\tilde{u} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.1)$$

In obtaining $\tilde{h}$ and $\tilde{u}$ we regard h and $\bar{u}$ as variables and H and U as constants. We may, however, reverse our point of view. That is, we regard H and U as variables and h and u as constants. In that case the right hand sides of (1.4) and (1.5) become functions of H and U, and from

$$\frac{dF}{dH}=0, \text{ and } \frac{d\phi}{dU}=0, \text{ we get}$$

$$h=\sqrt{\frac{n}{n-1}}\,\hat{H} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

$$u=\frac{n}{n-1}\hat{U} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.3)$$

The quantities $\hat{H}$ and $\hat{U}$ R. A. Fisher calls the *optimum* values.

3. *Moments*

Precision Constant

In order to distinguish between the moments of the two distributions treated in this paper, we shall denote the ith moment

of the precision constant about the origin by $\mu_i'(n)$ and that of its square by $\mu_i'(u)$ with similar notation for the moments about the mean.

Using the general definition of a moment of a continuous distribution, we obtain for the first moment of n, which is also its mean

$$\mu_1'(h) = \frac{n^{\frac{n-1}{2}} H^{n-1}}{2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty h^{-n} e^{-\frac{nH^2}{2h^2}} dh \quad \ldots\ldots (3.10)$$

To put this into an integrable form we make the transformation

$$t = \frac{nH^2}{2h^2} \quad \ldots\ldots\ldots\ldots (3.11)$$

This yields for the first moment

$$\mu_1'(h) = H\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots\ldots\ldots (3.12)$$

To facilitate calculation we express this in terms of factorials. For this purpose we have two cases to consider, namely, the case when n is even, and that when n is odd.

When n is even $\frac{n-2}{2}$ is an integer, and,

$$\mu_1'(h) = H\sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = H\sqrt{n} \frac{(n-4)(n-6)\cdots 2}{(n-3)(n-5)\cdots 1} \sqrt{\frac{2}{n}}$$

$$= H\sqrt{n} \frac{\left[2^{\frac{n-4}{2}} \left(\frac{n-4}{2}\right)!\right]^2}{(n-3)!} \sqrt{\frac{2}{n}} \quad \ldots\ldots (3.13)$$

When n is odd $\frac{n-1}{2}$ is an integer and hence

$$\mu_1'(h) = H\sqrt{\frac{n}{2}} \frac{\Gamma\frac{n-2}{2}}{\Gamma\frac{n-1}{2}} = H\sqrt{n} \frac{(n-4)(n-6)\cdots 1}{(n-3)(n-5)\cdots 2} \sqrt{\frac{n}{2}}$$

$$= H\sqrt{\pi}\,\frac{(n-3)}{\left[2^{\frac{n-3}{2}}\left(\frac{n-3}{2}\right)!\right]}\,2\sqrt{\frac{\pi}{2}} \qquad \ldots\ldots (3.14)$$

Similarly we obtain for the second, third, and forth moments of the distribution of about the origin the following expressions:

$$\mu_2'(h) = H^2\,\frac{n}{2}\,\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n}{n-3}\,H^2, \qquad \ldots\ldots\ldots (3.15)$$

$$\mu_3'(h) = H^3\left(\frac{n}{2}\right)^{3/2}\frac{\Gamma\left(\frac{n-4}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n}{n-4}\,\mu_1'(h)\,H^3, \qquad \ldots\ldots (3.16)$$

$$\mu_4'(h) = H^4\left(\frac{n}{2}\right)^2\,\frac{\Gamma\left(\frac{n-5}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n^2}{(n-3)(n-5)}\,H^4 \qquad \ldots\ldots (3.17)$$

Moments about the Mean

To study such properties of a distribution curve as skewness and flatness we must have the moments of the curve about the mean. To obtain these we use the well known formulae for expressing the moments about the mean in terms of the moments about any origin. Using these formulae we obtain for the first four moments of the precision constant, h, about the mean the following expressions:

$$\mu_1'(h) = 0$$

$$\mu_2(h) = \frac{\left[n-(n-3)\mu_1'^2(h)\right]H^2}{(n-3)} \qquad \ldots\ldots\ldots\ldots (3.18)$$

$$\mu_3(h) = \frac{\left[2(n-3)(n-4)\mu_1'^2 - n(2n-9)\mu_1'(n)\right]H^3}{(n-3)(n-4)} \qquad \ldots\ldots (3.19)$$

$$\mu_4(h) = \frac{[n^2(n-4)+2n(n-6)(n-5)\mu_1'^2 - 3(n-3)(n-4)(n-5)\mu'^4]H^4}{(n-3)(n-4)(n-5)} \quad \ldots\ldots (3.20)$$

where $\mu_1'(h)$ is given by (3.12).

For future use we shall give here approximations for $\mu_1'(h)$, $\mu_2(h)$, $\mu_3(h)$, and $\mu_4(h)$. The approximations were obtained by expanding the various quantities into power series of $\frac{1}{n}$. The derivation of these are not difficult but rather long and will therefore not be given in this paper.

These approximations are as follows:

$$\mu_1'(h) = \left(1+\frac{5}{4n}+\frac{49}{16n^2}\right)H \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.21)$$

$$\mu_2(h) = \frac{1}{2n} H^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.22)$$

$$\mu_3(h) = \frac{a}{n^2} H^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.23)$$

(where a is a constant)

$$\mu_4(h) = \frac{3}{4n^2} H^4 \quad \ldots\ldots (3.24)$$

Precision Constant Squared

The first moment of the precision constant squared distribution is, using the general definition of a continuous distribution curve about the origin

$$\mu_1'(u) = \frac{n^{\frac{n-1}{2}} U^{\frac{n-1}{2}}}{2^{\frac{n-3}{2}} \Gamma\frac{n-1}{2}} \int_0^\infty u^{-\frac{n+1}{2}} e^{-\frac{nU}{2u}} du \quad \ldots\ldots (3.30)$$

We reduce this to a known integral by means of the transformation

$$t = \frac{nU}{2u} \quad \ldots\ldots (3.31)$$

We then obtain for the first four moments of u about the origin the following simple expressions:

$$\mu_1'(u) = \frac{n}{n-3} U \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.32)$$

$$\mu_2'(u) = \frac{n^2}{(n-3)(n-5)} U^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.33)$$

$$\mu_3'(u) = \frac{n^3}{(n-3)(n-5)(n-7)} U^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.34)$$

$$\mu_4'(u) = \frac{n^4}{(n-3)(n-5)(n-7)(n-9)} U^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.35)$$

Moments about the Mean

For the moments about the mean of the precision constant squared distribution we have:

$$\mu_1 = 0$$

$$\mu_2 = \frac{2n^2}{(n-3)^2(n-5)} U \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.36)$$

$$\mu_3 = \frac{16n^3}{(n-3)^3(n-5)(n-7)} U^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.37)$$

$$\mu_4 = \frac{4n^4(2n+27)}{(n-3)^4(n-5)(n-7)(n-9)} U^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.38)$$

Skewness and Flatness

From the above expressions for the mean and also from the numerical values given in the tables we may conclude that the precision constant distribution is less skew than the distribution of the precision constant squared, at least for values of n up to 100. But what happens when n grows very large? To answer this we make use of the Pearsonian measure of skewness, β_1, defined by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \ldots\ldots\ldots\ldots \quad (3.40)$$

Since we are now interested in large values of n we make use of the approximate values of $\mu_2(h)$, and $\mu_3(h)$, which are

$$\mu_2(h) = \frac{1}{2n} H^2, \text{ and } \mu_3(h) = \frac{a}{n^2}$$

Hence we get

$$\lim_{n \doteq \infty} \beta_1(h) = \lim_{n \doteq \infty} \frac{a^2 H^6}{n^4} \div \frac{1}{n^3} H^6 = \lim_{n \doteq \infty} \frac{a^2}{n}$$

and $$\lim_{n \doteq \infty} \frac{a^2}{n} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.41)$$

To find $\lim_{n \doteq \infty} \beta_1(u)$ we make use of the exact values of $\mu_2(u)$ and $\mu_3(u)$ which are given by (3.36) and (3.37). This gives

$$\lim_{n \doteq \infty} \beta_1(u) = \lim_{n \doteq \infty} \left[\frac{16 n^3 U^3}{(n-3)^3(n-5)(n-7)}\right]^2 \div \left[\frac{2 n^2 U^2}{(n-3)^2(n-5)}\right]^3$$

$$= \lim_{n \doteq \infty} \frac{32(n-5)}{(n-7)^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.42)$$

From (3.41) and (3.42) we learn that both the precision constant distribution, and that of its square, approach perfect symmetry as the size of the sample, n, approaches infinity.

The flatness or kurtosis of a curve is measured by the quantity

$$\beta_2 = \frac{\mu_4}{\mu_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.43)$$

From this and the expressions (3.22 and (3.24), (3.33) and (3.35) we get

$$\lim_{n \doteq \infty} \beta_2(h) = \lim_{n \doteq \infty} \frac{3H^4}{4n^2} \div \left(\frac{H^2}{2n}\right)^2 = 3 \quad \ldots\ldots\ldots \quad (3.44)$$

and

$$\lim_{n \doteq \infty} \beta_2(u) = \lim_{n \doteq \infty} \frac{4n^4(2n+27)U^4}{(n-3)^4(n-5)(n-7)(n-9)} \div \frac{4n^4 U}{(n-3)^4(n-5)}$$

$$= \lim_{n=\infty} \frac{(n-5)(2n+27)}{(n-7)(n-9)} = 2 \quad \ldots\ldots\ldots \quad (3.45)$$

We may conclude, then, that while the distributions of the precision constant and its square are both perfectly symmetrical for very large values of n, they are nevertheless entirely distinct distribution curves for both small and large values of n, since $\lim_{n=\infty} \beta_2(h) = 3$, and $\lim_{n=\infty} \beta_2(\omega) = 2$. As a matter of fact the distribution of the precision constant approaches the normal curve, while the precision constant squared distribution approaches a curve of the form

$$y = y_0 \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}$$

where y_0 and a are constants.

4. *Quartiles and Medians*

The quartiles of a continuous distribution $f(x)$ may be defined by the equation

$$\int_0^{Q_i} f(x)\,dx = \frac{i}{4}, \quad (i = 1, 2, 3) \quad \ldots\ldots \quad (4.00)$$

For $i = 1$, Q_i is called the lower quartile; for $i = 2$, Q_i is called the median and for $i = 3$, Q_i is called the upper quartile.

In order to find the quartiles of the distributions studied in this paper we must make use of the incomplete Γ- function. This function is defined as follows

$$I(u,p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{1+p}} e^{-v} v^p\,dv \quad \ldots\ldots \quad (4.10)$$

Pearson's "Tables of the Inconplete Γ-Function" give the values of I, u, and p. Thus, if we know any two of the variables we can easily find the third.

Let us take, now, the distribution of h,

$$F(h)dh = \frac{n^{\frac{n-1}{2}} H^{n-1} h^{-n} e^{-\frac{nH^2}{2h^2}} dh}{2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right)}$$

The various quartiles of this distribution will be given by

$$n^{\frac{n-1}{2}} H^{n-1} \int_0^{Q_i} h^{-n} e^{-\frac{nH^2}{2h^2}} dh = \frac{i}{4} \quad \ldots . \quad (4.11)$$

By making the transformation

$$v = \frac{nH^2}{2h^2}$$

(4.11) is reduced to

$$\frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_{\infty}^{u_i \sqrt{\frac{n-1}{2}}} e^{-v} v^{\frac{n-3}{2}} = \frac{i}{4} \quad \ldots \ldots . \quad (4.13)$$

This is, of course, $I(u_i, \frac{u-3}{2})$, and since i and u are known we can easily find u_i from Pearson's Tables.

Comparing (4.11) and (4.13) and taking into account the transformation (4.12) we find that

$$Q_i = \frac{n^{\frac{i}{2}}}{u_i^{\frac{1}{2}} (2n-2)^{\frac{1}{4}}} H.$$

Since the lower limit in (4.13) is ∞ instead of 0 we see that the lower and upper quartiles are reversed.

To find the quartiles for the distribution of u we simply make use of the relation $u = h^2$.

In conclusion we may state that the results of this paper are similar to those of Professor Rietz on the distributions of the standard deviation and the variance.*

*Rietz, H. L. A comparison of the distributions curves of variance and of standard deviation, *Mathematical Monthly*, Vol. 36 (August-Sept., 1929), p. 355.

TABLE I

Modal and optimum values for the distribution of the precision constant and its square from a normal population for samples of 4, 10, 25, and 100.

	Pr. const. h		Pr. const. squared u	
n	Mode	Optimum	Mode	Optimum
4	H	$1.154H$	$0.800U$	$1.333U$
10	H	$1.054H$	$0.909U$	$1.111U$
25	H	$1.021H$	$0.962U$	$1.042U$
100	H	$1.005H$	$0.990U$	$1.010U$

TABLE II

Values of the mean and the first four moments about the mean of the distributions of the precision constant and its square for samples of 4, 10, 25, and 100 from a normal population.

Precision constant h				
n	Mean μ_1'	μ_2	μ_3	μ_4
4	$1.596H$	$1.454H^2$		
10	$1.153H$	$0.0982H^2$	$0.0678H^3$	$0.0815H^4$
25	$1.054H$	$0.0255H^2$	$0.0031H^3$	$0.0026H^4$
100	$1.013H$	$0.0053H^2$	$0.000141H^3$	$0.0000907H^4$
Precision constant square U				
n	μ_1'	μ_2	μ_3	μ_4
4	$4.000U$			
10	$1.429U$	$0.816U^2$	$3.110U^3$	$52.200U^4$
25	$1.136U$	$0.129U^2$	$0.06522U^3$	$0.892U^4$
100	$1.031U$	$0.224U^2$	$0.0199U^3$	$0.00129U^4$

TABLE III

Values of β_1 and β_2 for the distribution of the precision constant and its square for samples of 10, 25, 100, and ∞ from a normal population.

	h		u	
n	β_1	β_2	β_1	β_2
10	4.8548	8.4521	17.7807	78.3416
25	0.7596	3.9982	2.0099	5.3172
100	0.00134	3.2289	0.3515	2.5476
∞	0.0000	3.0000	0.0000	2.0000

TABLE IV

Medians and quartiles for the distributions of the precision constant and its square for samples of 4, 10, and 25 from a normal population.

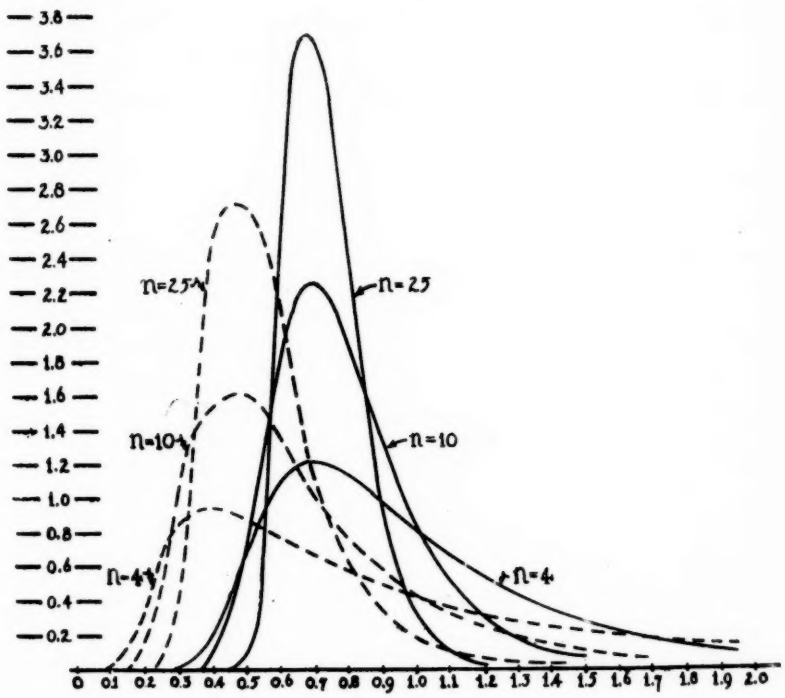
	Precision constant h			Precision constant squared u		
n	Q_1	Median	Q_3	Q_1	Median	Q_3
4	0.985 H	1.298 H	1.816 H	0.970 U	1.685 U	3.298 U
10	0.937 H	1.094 H	1.230 H	0.878 U	1.197 U	1.513 U
25	0.941 H	1.036 H	1.146 H	0.885 U	1.073 U	1.313 U

Distribution curves of the precision constant and its square for samples of 4, 10, and 25, from a normal population.

Note that the mode for the precision constant distribution is the same for all values of n as is seen from (2.0).

The solid curves are for the precision constant, the dotted curves for its square.

The unit used is the standard deviation of the population



A POSTULATE FOR OBSERVATIONS

By R. Henderson

When measurements are made by a given observer using a particular instrument, if the mean or expected result of the measurements is not supposed to be equal to the true value of the quantity measured the difference is considered to be an error, personal or instrumental or both. A correction is therefore applied to any such measurement in order to remove the discrepancy. In other words a given combination of observer and instrument is not considered to give correct or balanced measurements until this discrepancy is removed. Also as between two instruments or observers, both giving balanced measurements after the application of known corrections, preference is given to the one which shows the smaller variations between different measurements of the same quantity.

In selecting the formula to be used to determine, from the results of a series of measurements involving certain unknown quantities, the best measures of those quantities we are in a position similar to that of an observer desiring to make a certain measurement and selecting the best available instrument for the purpose. Such an observer would, in the first place, require that the instrument should give balanced measurements and would, in the second place, among a number of such instruments select the one showing the smallest standard deviation. This suggests the following definition and postulate.

Definition A balanced measure of a quantity is one of which the mean or expected value is equal to the true value of the quantity measured.

Postulate Of two or more balanced measures of a quantity the best measure is the one which has the smallest standard deviation.

Repeated Measurements

When we have a number of different results

$$a_1, a_2, a_3 \cdots \cdots a_n$$

of balanced measurements of the same quantity a, then any function of the form

$$\frac{l_1 a_1 + l_2 a_2 + \cdots + l_n a_n}{l_1 + l_2 + \cdots l_n}$$

will also be a balanced measure of the quantity. If the standard deviations of the individual measures are respectively

$$\sigma_1, \sigma_2 \cdots \cdots \sigma_n,$$

the square of the standard deviation of the function will be

$$\frac{l_1^2\sigma_1^2 + l_2^2\sigma_2^2 + \cdots \cdots l_n^2\sigma_n^2}{(l_1 + l_2 + \cdots l_n)^2}$$

This is a minimum when $l_1\sigma_1^2 = l_2\sigma_2^2 = \cdots \cdots = l_n\sigma_n^2$ so that the best measure is the average of the individual measures each weighted inversely as the square of its standard deviation. If the individual measures have the same standard deviation this reduces to the ordinary arithmetical average.

Combinations of Observations.

In applying this postulate to the theory of combination of observations, suppose that there are n unknown quantities

$x_i (i = 1, 2, 3 \cdots \cdots n$ and that we have balanced measures of

$m (m > n)$ linear functions of these unknowns of the form

$$y_h = \sum_{i=1}^{i=n} a_{hi} x_i \quad (h = 1, 2, 3 \cdots m) \qquad (1)$$

For simplicity we shall assume that the functions have been so taken that the standard deviations of these measures are all equal. Then a linear function of these measures of the form.

$$\sum_{h=1}^{h=m} b_{hj} y_h = \sum_{h=1}^{h=m} \sum_{i=1}^{i=n} b_{hj} a_{hi} x_i \qquad (2)$$

will be a balanced measure of x_j if

$$\sum_{h=1}^{h=m} b_{hj} a_{hi} = \delta_j^2 \qquad (3)$$

where $\delta_j^2 = 1$ if $i = j$ and $\delta_j^2 = 0$ if $i \neq j$

It will be the best measure of that type if $\sum_{h=1}^{h=m} b_{hj}^2$ is a minimum subject to those conditions.

By the method of indeterminate coefficients we find that this occurs when we can write

$$b_{hj} = \sum_{k=1}^{k=n} l_{jk} a_{hk} \qquad (4)$$

and the values of the n^2 coefficients l_{jk} are determined from the n^2 conditions (3). Then

$$x_j = \sum_{k=1}^{k=n} \sum_{h=1}^{h=m} l_{jk} a_{hk} y_h = \sum_{k=1}^{k=n} l_{jk} y_k \qquad (5)$$

it we write y_k for $\sum_{h=1}^{h=m} a_{hk} y_h$. The value of each of the n unknowns x_j is thus expressed in terms of the n func-

tions y_k or, in other words, may be determined from the n equations expressing y_k in terms of the n unknowns x_j. These equations take the form, if we write

$$A_{jk} \text{ for } \sum_{h=1}^{h=m} a_{hj}\, a_{hk}$$

$$\sum_{j=1}^{j=n} A_{jk}\, x_j = y_k \qquad (6)$$

It will be noted that although no assumption regarding the law of error, other than that of balance, has been made the equations deduced are the same as those derived, in the ordinary theory of least squares, from the assumption of the normal exponential law.

Measurement of Probabilities.

Where the quantity measured is a probability and the measure is to be determined from the observed result of a finite number of trials we know that, if the probability is p the number of trials n and the number of occurrences of the particular result r then the expected value of r is np. Consequently r/n is a balanced measure of p. The measure usually associated with Bayes' theorem, namely $\frac{r+1}{n+2}$ is not a balanced measure. Its mean value is $p + \frac{1-2p}{n+2}$ which is not equal to p unless p happens to be equal to ½.

For this case a different postulate might consistently with the general methods of science, have been proposed as follows. ***That hypothesis is to be adopted which makes the compound probability of the hypothesis and the observed facts a maximum.*** If then we considered one value of the probability as likely as another this would mean selecting the value of p which would make $p^r(1-p)^{n-r}$ a maximum. This would have given r/n as before.

Frequency Distributions

The notation on the subject of moments is so unsettled that it appears to be necessary for each writer to specify the notation adopted. In this paper the rth moment about the origin in a finite sample will be written m_r and the corresponding moment about the mean value will be designated by μ_r. The moments in the population from which the sample is drawn will be written $\bar{m}_r$ and $\bar{\mu}_r$ respectively.

In this connection an important consideration arises from the fact that balanced measures are not always consistent under ordinary mathematical transformations. This happens because if y is a balanced measure of x then $f(y)$ is not necessarily a balanced measure of $f(x)$ Let $y = x+h$ and let mean values be indicated by prefixing $\bar{m}_1$ so that

$$\bar{m}_1(h) = \bar{m}_1(y) - x = 0.$$

Then since

$$f(y) = f(h) + hf'(x) + \frac{h^2}{2} f''(x) + etc$$

we have

$$\bar{m}_1\{f(y)\} = f(x) + \frac{\bar{m}_1(h^2)}{2} f''(x) + c = f(x) + \frac{\bar{\mu}_2}{2} f''(x) + etc$$

Ordinarily therefore unless $f(x)$ is a linear function of x or, if not, y is an exact measure of x, $\bar{m}_1\{f(y)\}$ will not be equal to $f(x)$.

A simple illustration of this fact arises in connection with the determination, from a sample, of a measure for $\bar{\mu}_2$. By ordinary transformations we have the well known formula $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$ Also m_2 and m_1 are balanced measure of $\bar{m}_2$ and $\bar{m}_1$ respectively but m_1^2 is not a balanced measure of $\bar{m}_1^2$. We have in fact $\bar{m}_1(m_1^2) = \bar{m}_1^2 + \frac{1}{n}\bar{\mu}_2$. Therefore,

$$\bar{m}_1(\mu_2) = \bar{m}_1(m_2 - m_1^2) = \bar{m}_2 - \bar{m}_1^2 - \frac{1}{n}\bar{\mu}_2 = (1 - \frac{1}{n})\bar{\mu}_2.$$

The balanced measure of $\bar{\mu}_2$ would therefore be $\frac{n}{n-1}\mu_2$ which is not formally consistent with the balanced measures of

$\bar{m}_1$ and $\bar{m}_2$ in the light of the equation $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$. By a similar line of reasoning as shown by Thiele, we obtain $\frac{n^2}{(n-1)(n-2)}\mu_3$ as a balanced measure of $\bar{\mu}_3$ and by Tschuprow's modification of Thiele's analysis

$$\frac{n}{(n-1)(n-2)(n-3)}\{(n^2-2n+3)\mu_4 - 3(2n-3)\mu_2^2\}$$

as a balanced measure of $\bar{\mu}_4$. Here however we are faced with the further difficulty that while this is a balanced measure its standard deviation for small values of n is so great that possible values of μ_4 and μ_2 would result in negative values of $\bar{\mu}_4$ whereas in any real frequency distribution not only must $\bar{\mu}_4$ be positive but $(\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$ which is the mean value of $\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ must also be positive.

If, therefore, we wish to derive a value of $\bar{\mu}_4$ certainly satisfying this condition we must determine the average value of $\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ for all combinations of three values from the sample of n and use it as a balanced measure of $(\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$. This average value is found to be $\frac{n^2}{(n-1)(n-2)}(\mu_2\mu_4 - \mu_3^2 - \mu_2^3)$.

The analysis is as follows.

$$\bar{m}_1\left\{\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}\right\}$$

$$= \bar{m}_1\{x_1^4x_2^2 + 2x_1^3x_2^2x_3 - x_1^4x_2x_3 - x_1^3x_2^3 - x_1^2x_2^2x_3^2\}$$

$$= \bar{m}_4\bar{m}_2 + 2\bar{m}_3\bar{m}_2\bar{m}_1 - \bar{m}_4\bar{m}_1^2 - \bar{m}_3^2 - \bar{m}_2^3$$

$$= \bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3$$

If repetitions were allowed in the finite sample the average value would be the corresponding expression in moments of the sample, namely, $\mu_2\mu_4 - \mu_3^2 - \mu_2^3$. But since the expression vanishes if two or more of the values of x involved are equal the exclusion of repetitions reduces the total number of permutations three at a time from n^3 to $n(n-1)(n-2)$ without reducing the sum of the values. The average is thereincreased to

$$\frac{n^2}{(n-1)(n-2)}\left(\mu_2\mu_4 - \mu_3^2 - \mu_2^3\right)$$

The second moment $\bar{\mu}_2$ might have been similarly derived as the mean value of $\frac{(x_1 - x_2)^2}{2}$ and the third moment $\bar{\mu}_3$ as the mean value of

$$\frac{(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2)}{6}$$

In this latter case the expressions averaged do not vanish when two only of the value of x are equal but they cancel one another in pairs so that their sum vanishes.

We have thus as working approximations

$$\bar{\mu}_2 = \frac{n}{n-1}\mu_2$$

$$\bar{\mu}_3 = \frac{n^2}{(n-1)(n-2)}\mu_3$$

$$\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3 = \frac{n^2}{(n-1)(n-2)}\left(\mu_2\mu_4 - \mu_3^2 - \mu_2^3\right)$$

The net result of this investigation of the application of balanced measures as presumptive values of moments in frequency distributions seems to be that, in view of the formal inconsistencies involved, it is necessary to carefully select the functions to which such measures are applied. The functions considered above are suggested as well adopted for this purpose and

as probably sumcient for all practical purposes. If they are adopted as fundamental the resulting approximations for the Pearson constants $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ are

$$\bar{\beta}_1 = \bar{\mu}_3^2/\bar{\mu}_2^3 = \frac{n(n-1)}{(n-2)^2}\,\mu_3^2/\mu_2^3 = \frac{n(n-1)}{(n-2)^2}\,\beta_1 \qquad \text{and}$$

$$\bar{\beta}_2 - \bar{\beta}_1 - 1 = \frac{\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3}{\bar{\mu}_2^3} = \frac{(n-1)^2}{n(n-2)}\,\frac{\mu_2\mu_4 - \mu_3^2 - \mu_2^3}{\mu_2^3}$$

$$= \frac{(n-1)^2}{n(n-2)}\,(\beta_2 - \beta_1 - 1).$$

It will be noted that the coefficient in the latter equation is very nearly unity for even moderate values of n.

R Henderson

A SHORT METHOD FOR SOLVING FOR A COEFFICIENT OF MULTIPLE CORRELATION

By

PAUL HORST

The method which we present presupposes a familiarity with the Doolittle method [1] for solving normal equations. We start with the determinant

$$(1) \qquad R = \begin{vmatrix} 1 & r_{12} & \cdots & r_{1n} \\ r_{12} & 1 & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & & 1 \end{vmatrix}$$

where the elements are zero order coefficients of correlation.

Now the adjoint determinant of (1) may be written

$$(2) \qquad r = \begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{12} & R_{22} & \cdots & R_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{vmatrix}$$

where the elements are the cofactors of the elements in (1).

From the elementary theory of determinants [2] we know that

$$(3) \qquad r = R^{n-1}$$

The adjoint determinant of r may be designated by KR where

$$(4) \qquad KR = r^{n-1}$$

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[1] Mills, F. C., Statistical Methods, p. 577.

[2] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

From (3) and (4) we have

$$KR = R^{(n-1)^2}$$

or

$$K = R^{n(n-2)} \tag{5}$$

Hence the adjoint or r is obtained by multiplying each element of R by R^{n-2}. And if we write

$$KR = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix} \tag{6}$$

then

$$\frac{A_{ij}}{R^{n-2}} = r_{ij} \tag{7}$$

so that (1) may be rewritten

$$R = \begin{vmatrix} \frac{A_{11}}{R^{n-2}} & \frac{A_{12}}{R^{n-2}} & \cdots & \frac{A_{1n}}{R^{n-2}} \\ \frac{A_{12}}{R^{n-2}} & \frac{A_{22}}{R^{n-2}} & \cdots & \frac{A_{2n}}{R^{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{R^{n-2}} & \frac{A_{2n}}{R^{n-2}} & \cdots & \frac{A_{nn}}{R^{n-2}} \end{vmatrix} \tag{8}$$

The numerators of the elements in (8) are the cofactors of the elements in (2).

Let us now consider (8) as the coefficients of a set of normal equations whose constant terms are zero, and let us follow through literally the Doolittle elimination process.

For simplicity we outline the reduction of a 4-variable problem as follows.

Reciprocal	1	2	3	4	α	β	γ	δ
$-\frac{R^2}{A_{11}}$	$\frac{A_{11}}{R^2}$ -1	$\frac{A_{12}}{R^2}$ $\frac{-A_{12}}{A_{11}}$	$\frac{A_{13}}{R^2}$ $-\frac{A_{13}}{A_{11}}$	$\frac{A_{14}}{R^2}$ $\frac{-A_{14}}{A_{11}}$	α_1		γ_1	δ
		$\frac{A_{22}}{R^2}$ $-\frac{A_{12}^2}{R^2A_{11}}$	$\frac{A_{23}}{R^2}$ $-\frac{A_{12}A_{13}}{R^2A_{11}}$	$\frac{A_{24}}{R^2}$ $-\frac{A_{12}A_{14}}{R^2A_{11}}$	α_2	β_{22}		
$-\frac{R^2A_{11}}{AA_{1122}}$		$\frac{AA_{1122}}{R^2A_{11}}$ -1	$\frac{AA_{1123}}{R^2A_{11}}$ $-\frac{A_{1123}}{A_{1122}}$	$\frac{AA_{1124}}{R^2A_{11}}$ $-\frac{A_{1124}}{A_{1122}}$			γ_2	δ_2
			$\frac{A_{33}}{R^2}$ $-\frac{A_{13}^2}{R^2A_{11}}$ $-\frac{AA_{1123}^2}{R^2A_{11}A_{1122}}$	$\frac{A_{34}}{R^2}$ $-\frac{A_{13}A_{14}}{R^2A_{11}}$ $-\frac{AA_{1123}A_{1124}}{R^2A_{11}A_{1122}}$	α_3	β_{23} β_{33}		
$\frac{R^2A_{1122}}{AA_{112233}}$			$\frac{AA_{112233}}{R^2A_{1122}}$ -1	$\frac{AA_{112234}}{R^2A_{1122}}$ $\frac{A_{112234}}{A_{112233}}$			γ_3	δ_3
			$-\sum_2^n \beta_i n =$	$\frac{A'_{44}}{R^2}$ $-\frac{A_{14}^2}{R^2A_{11}}$ $-\frac{AA_{1124}^2}{R^2A_{11}A_{1122}}$ $-\frac{AA_{112234}^2}{R^2A_{1122}A_{112233}}$	α_4	β_{24} β_{34} β_{44}		
$\frac{R^2A_{112233}}{AA_{11223344}}$				$\frac{AA_{11223344}}{R^2A_{112233}}$ -1			γ_4	δ_4

The α-equations are the original equations with the coefficients to the left of the diagonal omitted. The β-equations are the product equations which are subtracted from the α-equations. The γ-equations are the reduced equations which may be represented symbolically by

$$\gamma = \alpha - \Sigma\beta \tag{9}$$

The δ-equations are the γ-equations divided by the negatives of their respective leading coefficients. That (9) is true may be readily proved from the theorem[3]

$$\begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} = AA_{ijkl} \tag{10}$$

Where the notation indicates cofactors rather than minors. The proof is made more obvious if (9) is written

$$\gamma = \left\{ \left[(\alpha - \beta_1) - \beta_2 \right] - \beta_3 \right\} - \beta_4 \text{ etc.}$$

for each successive subtraction reduces the determinant by one.

If we indicate the leading coefficients of the γ-equations by γ_{ii} we may prove that

$$R = \prod_1^n \gamma_{ii} \tag{11}$$

We have in the case of four variables

$$\prod_1^4 \gamma_{ii} = \frac{A_{11}}{R^2} \quad \frac{rA_{1122}}{R^2 A_{11}} \quad \frac{rA_{112233}}{R^2 A_{1122}} \quad \frac{rA_{11223344}}{R^2 A_{112233}} = \frac{r^3}{R^8} \tag{12}$$

$$A \equiv r$$

or from [3]

$$\prod_1^4 \gamma_{ii} = R$$

In the general case we have

$$\prod_1^n \gamma_{ii} = \frac{r^{n-1}}{R^{(n-2)n}} = \frac{R^{(n-1)^2}}{R^{n(n-2)}} = R$$

[3] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

Now let us consider Kelley's equation for the coefficient of multiple correlation[4] with a slight change in notation to be consistent with the above,

$$R_{n\cdot 12\cdots(n-1)} = \sqrt{1 - \frac{R}{R_{nn}}} \tag{13}$$

Obviously

$$R_{nn} = \prod_{1}^{n-1} \gamma_{ii} \tag{14}$$

So that (13) becomes simply

$$R_{n\cdot 12\cdots(n-1)} = \sqrt{1 - \gamma_{nn}} \tag{15}$$

But from (9) we have

$$\gamma_{nn} = \alpha_{nn} - \sum_{2}^{n} \beta_{in} \tag{16}$$

hence $$R_{n\cdot 12\cdots(n-1)} = \sqrt{1 - (\alpha_{nn} - \sum_{2}^{n} \beta_{in})}$$

But $\alpha_{nn} = 1$ therefore we get

$$R_{n\cdot 12\cdots(n-1)} = \sqrt{\sum_{2}^{n} \beta_{in}} \tag{17}$$

In other words $R_{n\cdot 12\cdots}$ is simply the square root of the last product summation.

From (17) it is obvious that the solution for the coefficient of multiple correlation is considerably shorter than the standard Doolittle solution for regression coefficients. All of the back solution work is eliminated, as is also the calculation of the last reciprocal.

The only caution needed with respect to the order of the variables is that the dependent variable shall be the nth variable.

The usual summation check method may be employed exactly as in the solution for regression coefficients.

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[4] Kelley, T. L., Statistical Method, p. 301, eq. 275.

Paul Horst

A SHORT METHOD AND TABLES FOR THE CALCULATION OF THE AVERAGE AND STANDARD DEVIATION OF LOGARITHMIC DISTRIBUTIONS*

By

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In fitting various types of curves to reaction-time data,[1] the writer was impressed with the enormous amount of labor and boredom involved in the calculation of the constants of logarithmic distributions. Besides the constant use of a set of logarithm tables, it requires the tedium of squaring large numbers on a machine to compute the second moments of the distributions. In order to eliminate some of the labor involved in such a process, a short method was devised for the computation of the average and the standard deviation of logarithmic distributions.

The short method described in this paper was originally developed to facilitate the work of fitting logarithmic normal curves to a large number of reaction-time distributions, but dispersions approximating this type seem to be sufficiently common in economics and biology to warrant a more general use of short methods in the computation of the constants of such distributions. In the field of economics, logarithmic curves have been fitted with success to distributions of income and prices, and probably could be applied equally well to distributions of capital. Many skewed distributions can also be found in the fields of biology and psychology. Kapteyn fitted a logarithmic curve to a distribution of the minimum weights necessary to produce a sensation of pres-

*A portion of the work involved in this paper was carried out during the writer's tenure as a National Research Fellow.

[1] Cf. Facilitation and Inhibition. *Arch. Psychol.* No. 86, 56 p.

sure.[2] Kapteyn attempts to show that logically the normal curve is the exception and skew curves the rule. For example, if "the diameters of certain ripe berries" are distributed in a normal curve, their *volumes* will be distributed in an asymmetrical curve; in other words, volume increase will be dependent upon size, so that volume changes are greater for large berries than for small ones. That skew curves are the rule can be shown analytically. Suppose certain quantities z are distributed normally, and *any* other quantities x are expressed as functions of z, thus,

$$z = f(x)$$

Then,

(1) $$dz = f'(x)dx$$

If the frequency curve for the z's is,

(2) $$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(z-M)^2}{2\sigma^2}}$$

then the frequency curve for the x's is,

(3) $$y \frac{N}{\sigma\sqrt{2\pi}} f'(x)\, e^{-\frac{[f(x)-M]^2}{2\sigma^2}}$$

It will be seen at once that the x's cannot be distributed normally provided x is a non-linear function of z. If we let $z = \log x$, then $dz = \frac{dx}{x}$ and equation (2) becomes

(4) $$y = \frac{N}{\sigma\sqrt{2\pi}} \frac{1}{x} e^{-\frac{(\log x - M)^2}{2\sigma^2}}$$

This is the logarithmic curve of distribution, the theory of which has been treated by several writers, one of the first and most important papers on this subject being that of McAllister.[3] The study

[2] J. C. Kapteyn, "Skew frequency curves in Biology and Statistics". Groningen. p. 42-43, 1903.

[3] The Law of the Geometric Mean. *Proc. Roy. Soc.* 29:367. (1879).

of the properties of the logarithmic curve of error was undertaken by McAllister at the suggestion of Galton,[4] who saw the possibility of applying it to psychological and social phenomena.

Dispersions approximating this type are illustrated by distributions which are definitely limited at the zero point, but a more definite presumption in favor of the logarithmic curve is indicated when the *real* origin, determined *a priori* or deduced from empirical considerations, does not correspond with the origin on the value scale.

Reaction-time distributions are good examples of dispersions where a displacement of the origin is indicated by empirical considerations. A little reflection will show that there must be a physiological limit for the speed of reaction. It takes a certain minimum time for the neuro-muscular machine to do its work. The time it takes for the machine to do its work constitutes an undisturbed region within which no deviations ever occur. Reaction-time dispersions approximate the logarithmic more closely than the normal curve of error. Investigations in the field of learning often give distributions which have origins other than the zero of the scale which can be determined *a priori* . . . .that is. the real origin follows *inevitably* from the conditions of the experiment. If the norm of mastery for learning a maze is two perfect trials out of three, then the criterion is such that an animal to learn a maze must make at least two perfect runs. In other words, the experimenter's criterion is such that no deviations could *possibly* occur under two trials.

In using the short method for finding the first and second moments of a logarithmic distribution, the computer must still resort to a table, but in this case it is only necessary to use a single page table instead of an extensive logarithm table. Furthermore, the labor of squaring the logarithms is eliminated. The short method can best be explained by following the process through an

[4] The Geometric Mean in Vital and Social Statistics. *Proc. Roy. Soc.* 29:365. (1879).

actual example.[5] This is illustrated in Table II on a distribution of reaction-times. Beginning at 70 (the real origin of the distribution is assumed to be at 70) the step-intervals are numbered from zero to the end of the distribution. Under the log x column of Table I the value for each step is found and multiplied by the frequency for each step. This operation gives the values shown in the $F \log x$ column of Table II. The sum of these values divided by N (number of cases) gives the correction C. The average ($\log G_l$) for the logarithmic distribution is finally found by adding a factor K to the correction C. The constant K depends upon the length of the step-interval. In this distribution, the length of the step-interval is ten. Looking under column K of Table I, we find that the value of K for a step-interval of ten units is equal to .69897. The geometric mean (G) of the distribution is found by adding 70 to (G_l).[6]

The process of finding the second moment and standard deviation (σ_g) is similar to that for finding the first moment and the average. In one respect it is simpler: no correction has to be added for the length of step. The $F \log^2 x$ column is obtained by multiplying the value for each step in the $\log^2 x$ column of Table I by its appropriate frequency. The sum of these divided by N (number of cases) gives the crude unit moment. The square of the correction C is then subtracted to give the corrected unit moment around the average. The square root of the corrected unit moment around the average gives the standard deviation (σ_g), and the antilog of σ_g gives the standard deviation ratio (σ_r).

The formula for finding the average of a logarithmic distribution is,

$$\log G_l = \frac{\Sigma F \log x}{N} + K = C + K$$

[5] For those interested, the proof of the formulae for getting the average and standard deviation is given in an appendix at the end of this paper.

[6] One would expect the geometric mean to be different if the origin were taken at a point other than 70. An origin at 70 was assumed because it results in an extremely good fit to the distribution. In this case, 70 would correspond to the physiological limit below which no deviations ever occur.

TABLE II

Time	F	Step	$F \log x$	$F \log^2 x$
70	1	1		
80	3	2	1.431363	.682934
90	14	3	9.785580	6.839826
100	40	4	33.803921	28.567627
10	55	5	52.483338	50.081832
20	60	6	62.483561	65.069923
30	52	7	57.925054	64.525229
40	46	8	54.100197	63.626769
50	33	9	40.604814	49.962150
60	28	10	35.805100	45.785901
70	21	11	27.766605	36.713541
80	14	12	19.064189	25.960237
90	9	13	12.581460	17.588126
200	7	14	10.019546	14.341615
10	7	15	10.236785	14.970255
20	4	16	5.965446	8.896638
30	3	17	4.555541	6.917653
40	1	18	1.544068	2.384146
50				
60	1	20	1.591064	2.531486
70				
\80	1	22	1.633468	2.668219
	400		400)443.381100	400)508.114107
				1.270285
			$C =$ 1.108452	$C^2 =$ 1.228667
			$K =$ .698970	
				$\sigma_g^2 =$ .041618
			log $G_1 =$ 1.807422	
				$\sigma_g =$ .204004
For origin at 70			$G_1 =$ 65.8	
			$G =$ 135.8	$\sigma_r =$ 1.59

TABLE I

Step	Log x	$(\text{Log } x)^2$	Step Interval	K
1	.00000 00000	.00000 00000	1	.30102 99956
2	.47712 12547	.22764 46917	2	.00000 00000
3	.69897 00043	.48855 90669	3	.17609 12590
4	.84509 80400	.71419 06972	4	.30102 99956
5	.95424 25094	.91057 87668	5	.39794 00086
6	1.04139 26851	1.08449 87247	6	.47712 12547
7	1.11394 33523	1.24086 97921	7	.54406 80443
8	1.17609 12590	1.38319 06496	8	.60205 99913
9	1.23044 89213	1.51400 45481	9	.64321 25137
10	1.27875 36009	1.63521 07719	10	.69897 00043
11	1.32221 92947	1.74826 38633	11	.74036 26894
12	1.36172 78360	1.85430 26993	12	.77815 12503
13	1.39794 00086	1.95423 62678	13	.81291 33566
14	1.43136 37641	2.04880 22253	14	.84509 80400
15	1.46239 79978	2.13860 79042	15	.87506 12633
16	1.49136 16938	2.22415 97018	16	.90308 99869
17	1.51851 39398	2.30588 45856	17	.92941 89257
18	1.54406 80443	2.38414 61255	18	.95424 25094
19	1.56820 17240	2.45925 66473	19	.97772 36052
20	1.59106 46070	2,53148 65837	20	1.00000 00000
21	1.61278 38567	2.60107 17684	21	1.02118 92990
22	1.63346 84555	2.66821 91953	22	1.04139 26851
23	1.65321 25137	2.73311 16157	23	1.06069 78403
24	1.67209 78579	2.79591 12465	24	1.07918 12460
25	1.69019 60800	2.85676 27889	25	1.09691 00130
26	1.70757 01760	2.91579 59062	26	1.11394 33523
27	1.72427 58696	2.97312 72744	27	1.13033 37684
28	1.74036 26894	3.02886 22909	28	1.14612 80356
29	1.75587 48556	3.08309 65087	29	1.16136 80022
30	1.77085 20116	3.13591 68471	30	1.17609 12590
31	1.78532 98350	3.18740 26197	31	1.19033 16981
32	1.79934 05494	3.23762 64129	32	1.20411 99826
33	1.81291 33566	3.28665 48386	33	1.21748 39442
34	1.82607 48027	3.33454 91850	34	1.23044 89213
35	1.83884 90907	3.38136 59785	35	1.24303 80486
36	1.85125 83487	3.42715 74737	36	1.25527 25051

37	1.86332	28601	3.47197	20810	37	1.26717 17284
38	1.87506	12633	3.51585	47414	38	1.27875 36009
39	1.88649	07251	3.55884	72561	39	1,29003 46113
40	1.89762	70912	3.60098	85775	40	1.30102 99956
41	1.90848	50188	3.64231	50672	41	1.31175 38610
42	1.91907	80923	3.68286	07246	42	1.32221 92947
43	1.92941	89257	3.72265	73909	43	1.33243 84599
44	1.93951	92526	3.76173	49312	44	1.34242 26808
45	1.94939	00066	3.80012	13980	45	1.35218 25181
46	1.95904	13923	3.83784	31768	46	1.36172 78360
47	1.96848	29485	3.87492	51187	47	1.37106 78622
48	1.97772	36052	3.91139	06589	48	1.38021 12417
49	1.98677	17342	3.94726	19240	49	1.38916 60843
50	1.99563	51945	3.98255	98299	50	1.39794 00086

and, $$G_1 = \text{antilog}(C + K)$$

where G_1 is the geometric mean measured from an origin which may be other than the zero of the scale. The geometric mean (G) measured from the zero of the value scale is,

$$G = G_1 + (\text{displacement of the origin})$$

The formula for finding the standard deviation around the average is,

$$\sigma_g = \sqrt{\frac{\Sigma F \log^2 x}{N} - C^2}$$

and, $$\sigma_r = \text{antilog } \sigma_g$$

Summary of steps in the calculation of the average ($\log G_1$) by the short method:

1. Beginning at the origin, find the deviation of the mid-point of each step-interval from the origin in units of step-interval.
2. Using Table I, find the $\log x$ of each step-deviation and weight it by its appropriate F (frequency).

3. Find the *sum* of the $F\log^2 x$'s, and divide this sum by N (number of cases). This gives the *correction* C.
4. Using Table I, find the value of K corresponding to the number of units in the step-interval. Add the factor K to the correction C to get the average $(\log G_1)$.

Summary of steps in the calculation of the standard deviation (σ_g) around the average $(\log G_1)$ by the short method:

1. Using Table 1, find the $\log^2 x$ of each step-deviation and weight it by its appropriate frequency.
2. Find the *sum* of the $F \log^2 x$'s, and divide this sum by N.
3. Then subtract the square of the correction C to get the *second unit moment* around the average.
4. Extract the square root of the second unit moment to obtain the standard deviation (σ_g).

APPENDIX

Deduction of the Formulae for the Short Method

Let $\log G$ be the logarithmic mean, x, the length of step, $f_1 \ldots\ldots f_n$ the frequencies for successive steps, and $m_1, m_2 \ldots\ldots m_n$ the mid-points of the steps for origin at zero. Then the first mid-point, m_1, is at $x/2$ the second, m_2, is at $3x/2$ etc. For convenience, these items may be arranged in the form of a table.

Midpoint (M)	F	$F \log M$	$F \log^2 M$
m_1	f_1	$f_1 \log x/2$	$f_1 \log^2 x/2$
m_2	f_2	$f_2 \log 3x/2$	$f_2 \log$
.	.	.	
.	.	.	
.	.	.	
m_n	f_n	$f_n \log \frac{2n-1}{2} x$	$f_n \log^2 \frac{2n-1}{2} x$

$$N \log G = \frac{\Sigma f_n \log \frac{2n-1}{2} x}{N}, \quad \sigma_g^2 = \frac{\Sigma f_n \log^2 \frac{2n-1}{2} x}{N} - \log^2 G$$

where G is the geometric mean, and σ_g is the standard deviation around $\log G$.

We have, therefore,

$$\log G = \frac{f_1 \log {}^{x}/_{2} + f_1 \log {}^{3x}/_{2} + \cdots\cdots\cdots + f_n \log \left(\frac{2n-1}{2}\right)x}{N}$$

Stating the logarithm of each fraction as the sum or difference of the logarithms of its factors, we have,

$$\log G = \frac{f_1 (\log 1 - \log 2 + \log x) + \cdots + f_n \left[\log(2n-1) - \log 2 + \log x\right]}{N}$$

$$= \frac{f_1 \log 1 + f_2 \log 3 + \cdots + f_n \log(2n-1)}{N} + \log x - \log 2$$

$$= \frac{\Sigma \left[f_n \log (2n-1)\right]}{N} + \log x - \log 2$$

Since

$$\log x = \frac{\Sigma f \log x}{N}$$

and,

$$\log 2 = \frac{\Sigma f \log 2}{N}$$

Letting

$$K = \log x - \log 2$$

and,

$$C = \frac{\Sigma \left[f_n \log (2n-1)\right]}{N}$$

we finally have,

$$\log G = C + K$$

Where C is the correction, and K is the constant indicated in Table I.

For the second unit moment around $\log G$, we have,

$$\sigma_g^2 = \frac{\Sigma\left[f_n \log^2\left(\frac{2n-1}{2}\right)x\right]}{N} - \left(\frac{\Sigma\left[f_n \log\left(\frac{2n-1}{2}\right)x\right]}{N}\right)^2$$

$$= \frac{\Sigma\left\{f_n\left[\log^2\left(\frac{2n-1}{2}\right) + 2\log\left(\frac{2n-1}{2}\right)\log x + \log^2 x\right]\right\}}{N}$$

$$- \left\{\left(\frac{\Sigma f_n \log\left(\frac{2n-1}{2}\right)}{N}\right)^2 + \frac{2\Sigma\left[f_n \log\left(\frac{2n-1}{2}\right)\right]\log x}{N} + \log^2 x\right\}$$

Expanding again and collecting terms, we have,

$$\sigma_g^2 = \frac{\Sigma\left\{f_n\left[\log(2n-1)\right]^2\right\}}{N} - \left\{\frac{\Sigma\left[f_n \log(2n-1)\right]}{N}\right\}^2$$

$$= \frac{\Sigma\left\{f_n\left[\log(2n-1)\right]^2\right\}}{N} - C^2$$

In Table I, $x = (2n-1)$ so that,

$$\log x = \log(2n-1)$$

and $$(\log x)^2 = \left[\log(2n-1)\right]^2$$

For each step, 1, 2, 3 . . . . n, the corresponding values of x are 1, 3, 5 . . . . $2n-1$. Note that x as used in the table is not the same as x as used in the deduction of the formulae.

The figures in Table I are accurate to ten places of decimals.

The log x column consists simply of the logarithms of odd numbers from one to one hundred. K was computed by subtracting log 2 from the logarithm of each number indicated in the "step interval" column. The $(\log x)^2$ column was computed by squaring fifteen place logarithms with the aid of calculating machines. This had to be done by indirect methods through the use of the simple algebraic relationship, $(a+r)^2 = a^2 + 2ar + r^2$, where a is the first part of the number and is the remainder. The table was computed by two different persons and checked on two different calculating machines by each person.

ON SYMMETRIC FUNCTIONS OF MORE THAN ONE VARIABLE AND OF FREQUENCY FUNCTIONS

By

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In a paper published in this journal[1] the writer has developed a simple differential operator method for expressing any symmetric function of the n variates $x_1, x_2, \cdots, x_n$ as a rational, integral, algebraic function of the power sums $s_1, s_2, \cdots, s_w$ where w is the weight of the symmetric function and

$$s_k = \Sigma x_i^k = x_1^k + x_2^k + \cdots + x_n^k.$$

The transformation to moments is then simply a matter of recognizing that $nu'_k = s_k$ if u'_k is the kth moment of the n variates with respect to the origin from which they are measured. If the origin is at the arithmetic mean of the n variates the prime may be dropped and then $nu_k = s_k$.

In the above mentioned paper the variates x_i are of the serial distribution type, but, of course, not necessarily integers. The extensions to the case of more than one set of variates and to frequency functions now suggest themselves. It is the purpose of this note to discuss these problems simultaneously.

Suppose that two sets, of n variates each, $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots y_n$ are given and that x_i, y_i $(i = 1,2, \ldots, n)$ are corresponding pairs. Modifying the partition notation used in the previous paper the symmetric function to be considered may be written in the form $(a_1^{m_1} a_2^{m_2} a_3^{m_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$ i.e. the sum of all such terms as

$$x_1^{a_1} x_2^{a_1} \cdots x_{n_1}^{a_1} x_{n_1+1}^{a_2} \cdots x_{n_1+n_2}^{a_2} \cdots y_1^{b_1} y_2^{b_1} \cdots y_{m_1}^{b_1} y_{m_1+1}^{b_2} \cdots y_{m_1+m_2}^{b_2} \cdots$$

[1] Symmetric Functions and Symmetric Functions of Symmetric Functions, Vol. II. No. 2 (May, 1931), pp. 102-149.

where

$$a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots n_1, n_2, n_3, \cdots m_1, m_2, m_3, \cdots$$

are positive integers and where

$$a_1 > a_2 > a_3 > \cdots > 0.$$

$$n_1 + n_2 + n_3 + \cdots = m_1 + m_2 + m_3 + \cdots$$

e.g. $(3^2.451) = (332.451) = \sum x_i^3 x_j^3 x_k^2 y_i^4 y_j^5 y_k$, $i \neq j \neq k$. Obviously the order of the parts in the bipartite notation will be important, corresponding parts being equidistant from the left of their respective sections of the partition. The double partition will be said to be of weight $w_1 = a_1 n_1 + a_2 n_2 + a_3 n_3 + \cdots$ in x and of weight $w_2 = b_1 m_1 + b_2 m_2 + b_3 m_3 + \cdots$ in y. There will be no loss of generality if it is assumed that $w_1 \geq w_2$.

By a procedure similar to that employed in the first chapter of the paper already referred to, it may be shown that any symmetric function of the type defined above can be expressed as rational, integral, algebraic function of the symmetric functions

$$s_{11}, s_{12}, s_{21}, \cdots\cdots, s_{w_1 w_2}$$

where

$$s_{hk} = \sum_{i=1}^{n} x_i^h y_i^k, \quad h = 1, 2, \cdots, w_1, \quad k = 1, 2, \cdots, w_2.$$

Moreover, the terms of this function will be isobaric of weight w_1 in x, isobaric of weight w_2 in y, and hence isobaric of weight $w_1 + w_2$ in x and y together.

e. g. Multiply s_{21} by itself:

$$s_{21}^2 = (x_1^2 y_1 + x_2^2 y_2 + \cdots + x_n^2 y_n)^2$$

$$= (x_1^4 y_1^2 + x_2^4 y_2^2 + \cdots + x_n^4 y_n^2) + 2(x_1^2 x_2^2 y_1 y_2 + x_1^2 x_3^2 y_1 y_3 + \cdots)$$

$$= s_{42} + 2(2^2.1^2).$$

$$(2^2.1^2) = (s_{21}^2 - s_{42})/2,$$

each term being of weight 4 in x, 2 in y, and 6 in x and y together. It is possible then to write

$$(a_1^{m_1}a_2^{m_2}a_3^{m_3}\cdots\, b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)\equiv$$

$$f(s_{11}, s_{12}, \cdots, s_{ij}, \cdots, s_{w_1 w_2}) \equiv f$$

where f stands for a rational, integral, algebraic function of the sums (or product moments) s_{ij}, $i=1,\cdots,w_1$, $j=1,\cdots,w_2$, isobaric as explained above. Suppose that a new pair of variates $x_{n+1}=\alpha$, $y_{n+1}=\beta$ are introduced. Obviously s_{ij} becomes $s_{ij}+\alpha^i\beta^j$. Hence applying Taylor's Theorem f becomes

$$f+(\alpha\beta d_{11}+\alpha\beta^2 d_{12}+\cdots+\alpha^i\beta^j d_{ij}+\cdots+\alpha^{w_1}\beta^{w_2}d_{w_1 w_2})f$$

$$+(\alpha\beta d_{11}+\alpha\beta^2 d_{12}+\cdots+\alpha^i\beta^j d_{ij}+\cdots+\alpha^{w_1}\beta^{w_2}d_{w_1 w_2})^2 f/2!$$

$$+\cdots\cdots\cdots$$

$$+(\alpha\beta d_{11}+\alpha\beta^2 d_{12}+\cdots+\alpha^i\beta^j d_{ij}+\cdots+\alpha^{w_1}\beta^{w_2}d_{w_1 w_2})^{w_2} f/w_2!$$

all other terms being identically zero, where $d_{ij}=\partial/\partial_{ij}$ and $d_{ij}^k=\partial^k/\partial s_{ij}^k$, $i=1,\cdots,w_1$, $j=1,\cdots,w_2$.

Using the multinomial theorem and collecting coefficients of $\alpha^i\beta^j$ the above expression may be written in the form

$$(1+\alpha\beta D_{11}+\alpha\beta^2 D_{12}+\cdots+\alpha^i\beta^j D_{ij}+\cdots)f$$

where

$$(1)\quad \begin{cases} D_{11}=d_{11}, \\ D_{12}=d_{12}, \\ D_{21}=d_{21} \end{cases}$$

$$\begin{cases} D_{13}=d_{13} \\ D_{22}=d_{22}+d_{11}^2/2, \\ D_{31}=d_{31}, \end{cases}$$

$$\begin{cases} D_{14}=d_{14}, \\ D_{23}=d_{23}+d_{11}d_{12}, \\ D_{32}=d_{32}+d_{11}d_{21}, \\ D_{41}=d_{41}, \end{cases}$$

$$\left\{\begin{array}{l} \left[\begin{array}{l} D_{15} = d_{15}, \\ D_{24} = d_{24} + d_{11}d_{13} + d_{12}^2/2, \\ D_{33} = d_{33} + d_{11}d_{22} + d_{12}d_{21} + d_{11}^3/6, \\ D_{42} = d_{42} + d_{11}d_{31} + d_{21}^2/2, \\ D_{51} = d_{51} \end{array}\right. \\ \\ D_{ij} = \sum \dfrac{d_{i_1 j_1}^{k_1} \; d_{i_2 j_2}^{k_2} \; d_{i_3 j_3}^{k_3} \cdots}{k_1! \quad k_2! \quad k_3! \cdots}, \quad i = 1, \cdots, w_1, \; j = 1, \cdots, w_2, \end{array}\right.$$

where $k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots = i,$
$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots = j,$

$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers. The inverse relation is given bv

$$d_{ij} = \sum \frac{(-1)^{v+1}(v-1)! D_{i_1 j_1}^{k_1} \; D_{i_2 j_2}^{k_2} \; D_{i_3 j_3}^{k_3} \cdots}{k_1! \quad k_2! \quad k_3! \cdots},$$

where $i = 1, \cdots, w_1, \quad j = 1, \cdots, w_2,$

$$k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots = i,$$

$$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots = j,$$

$$k_1 + k_2 + k_3 + \cdots = v,$$

$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers.

The effect of the new variates α, β on

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

is to replace this symmetric function by

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$+ \alpha^{a_1} \beta^{b_1} (a_1^{n_1 - 1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1 - 1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$+ \alpha^{a_2} \beta^{b_2} (a_1^{n_1} a_2^{n_2 - 1} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2 - 1} b_3^{m_3} \cdots) + \cdots.$$

Hence replacing f by $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$, then

$$(1 + \alpha\beta D_{11} + \alpha\beta^2 D_{12} + \alpha^2\beta D_{21} + \cdots$$

$$+ \alpha^i \beta^j D_{ij} + \cdots)(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$= (a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$+ \alpha^{a_1} \beta^{b_1} (a_1^{n_1 - 1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1 - 1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$+ \alpha^{a_2} \beta^{b_2} (a_1^{n_1} a_2^{n_2 - 1} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2 - 1} b_3^{m_3} \cdots) + \cdots.$$

Equating coefficients of like terms in α and β it is seen that

$$(2) \quad \begin{cases} D_{a_1 b_1} (a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots) \\ = (a_1^{n_1 - 1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1 - 1} b_2^{m_2} b_3^{m_3} \cdots), \\ D_{a_2 b_2} (a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots) \\ = (a_1^{n_1} a_2^{n_2 - 1} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2 - 1} b_3^{m_3} \cdots), \\ \text{etc. and} \\ D_{hk} (a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots) = 0 \quad \text{if both } h \text{ and } k \text{ are not among } a_1, a_2, a_3, \cdots \text{ and } b_1, b_2, b_3, \cdots \text{ respectively.} \end{cases}$$

Hence also

$$D_{a_1 b_1}^{t_1} D_{a_2 b_2}^{t_2} D_{a_3 b_3}^{t_3} \cdots D_{a_k b_k}^{t_k} (a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots) = 0$$

if $t_1, t_2, t_3, \cdots, t_k$ are not all respectively less than or equal to $n_1, n_2, n_3, \cdots, n_k$ and also respectively less than or equal to $m_1, m_2, m_3, \cdots, m_k$ where $t_1, t_2, t_3, \cdots, t_k$ are positive integers or zeros; that is, if even one of the values of t is greater than the corresponding n or m then the effect of the operation is to give zero. This last property of the D operators is important not only in minimizing the work of computation as will be seen in the illustrations given below, but will be of fundamental importance in the theorem to be stated in the closing paragraph of this paper. It should be noted here that the multiplication of the operators is commutative.

To illustrate the application of the operators consider:

a) $(21.1^2) = c_1 s_{21} s_{11} + c_2 s_{32}$ where c_1 and c_2 are constants to be determined. These are the only terms which satisfy the weight conditions. Operating on the left with D_{21} and on the right with

d_{21} gives (1.1) $= c_1 s_{11}$, i.e. $s_{11} = c_1 s_{11}$, hence $c_1 = 1$.
Operating on the left with D_{32} and on the right with $d_{32} + d_{11} d_{21}$ gives $0 = c_1 + c_2$, hence $c_2 = -1$ and thus $(21.1^2) = s_{21} s_{11} - s_{32}$.

b) $(321.1^3) = c_1 s_{63} + c_2 s_{52} s_{11} + c_3 s_{51} s_{12} + c_4 s_{42} s_{21} + c_5 s_{41} s_{22}$
$+ c_6 s_{41} s_{11}^2 + c_7 s_{32} s_{31} + c_8 s_{31} s_{21} s_{11} + c_9 s_{21}^3$.

These are the only terms which satisfy the weight conditions. If the expressions for all the symmetric functions of lower weights are known then to determine the constants it is sufficient to operate on the left with

$D_{11}, D_{21}, D_{31}, D_{41}, D_{31}, D_{63}$ and on the right with their respective equivalents in terms of d. However if the expressions for the symmetric functions of lower weights are not known then it is perhaps simpler to operate with

$d_{12}, d_{11}^2, d_{21}^2, \quad d_{31} d_{21} d_{11}, \quad d_{31} d_{32}, d_{41} d_{22}, d_{42} d_{21}$,
$d_{11} d_{52}, d_{63}$ on the right and with their respective equivalents in terms of D on the left. d_{12} and D_{12} give $0 = c_3 s_{51}$ and hence $c_3 = 0$. d_{11}^2 and D_{11}^2 give $0 = 2 c_6 s_{41}$ and $c_6 = 0$. d_{21}^2 and D_{21}^2 give $0 = 6 c_9 s_{21}$ and $c_9 = 0$. Similarly $d_{31} d_{21} d_{11}$ and $D_{31} D_{21} D_{11}$ give $c_8 = 1$. $d_{31} d_{32}$ and $D_{31}(D_{32} - D_{21} D_{11})$ give $c_7 = -1$. $d_{41} d_{22}$ and $D_{41}(D_{22} - D_{11}^2/2)$ give $c_5 = 0$. $d_{21} d_{42}$ and $D_{21}(D_{42} - D_{31} D_{11} - D_{21}^2/2)$ give $c_4 = -1$. $d_{11} d_{52}$ and $D_{11}(D_{52} - D_{41} D_{11} - D_{31} D_{21})$ give $c_2 = -1$. d_{63} and $D_{63} - D_{52} D_{11} - D_{51} D_{12} - D_{42} D_{21} - D_{41} D_{22} + D_{41} D_{11}^2 - D_{32} D_{31} + 2 D_{31} D_{21} D_{11} + D_{21}^3/2$ give $c_1 = 2$. Hence $(321.1^3) = 2 s_{63} - s_{52} s_{11} - s_{42} s_{21} - s_{32} s_{31} + s_{31} s_{21} s_{11}$.

Suppose that $y_k = 1$, $k = 1, 2, \cdots, n$. Then s_{ij} is simply s_i and D_{ij}, d_{ij} have no meaning except when $i = j$ and then D_{ii} and d_{ii} become the operators D_i and d_i respectively, of the earlier paper.

The operator relations for any number of sets of corresponding variates are now obvious. For instance, in the case of 3 sets

x_i, y_i, z_i the result is

$$D_{ijk} = \sum \frac{d_{i_1 j_1 k_1}^{h_1}}{h_1!} \frac{d_{i_2 j_2 k_2}^{h_2}}{h_2!} \frac{d_{i_3 j_3 k_3}^{h_3}}{h_3!} \cdots ,$$

$$i = 1,2, \cdots, w_1,$$

$$j = 1,2, \cdots, w_2,$$

$$k = 1,2, \cdots, w_3,$$

$$h_1 i_1 + h_2 i_2 + h_3 i_3 + \cdots = i,$$

$$h_1 j_1 + h_2 j_2 + h_3 j_3 + \cdots = j,$$

$$h_1 k_1 + h_2 k_2 + h_3 k_3 + \cdots = k,$$

$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots, h_1, h_2, h_3, \cdots$ being positive integers, w_1, w_2, w_3 the weights in xyz respectively of the symmetric function,

$$d_{ijk} = \partial / \partial s_{ijk}, \quad s_{ijk} = \sum_{t=1}^{n} x_t^i y_t^j z_t^k .$$

Returning now to the case of two variates x and y, suppose that x_i takes on only integral values for $i = 1,2, \ldots, n$ and that $y_i = f(x_i)$ where $f(x_i)$ is thought of as the frequency corresponding to $x = x_i$. If, further, $b_1 = b_2 = b_3 = \cdots = 1$ then the operators developed in this note give the expressions for $(a^\alpha b^\beta c^\gamma \ldots)$ of the earlier paper when each serial x_i^k is replaced by $x_i^k f(x_i)$. More generally, let $y_i = f(x_i)\Delta x_i$ represent the frequency of x in the interval Δx_i. If x takes on only integral values then of course $\Delta x_i = 1$, $i = 1,2, \ldots, n$. Consider

$$(32.11) = \sum_{\substack{i=1 \\ j=1}}^{n} x_i^3 x_j^2 f(x_i) f(x_j) \Delta x_i \Delta x_j, \quad i \neq j,$$

$$= \sum_{i=1}^{n} x_i^3 f(x_i) \Delta x_i \cdot \sum_{j=1}^{n} x_j^2 f(x_j) \Delta x_j - \sum_{i=1}^{n} x_i^5 f^2(x_i) \overline{\Delta x_i}^2 .$$

If the lower and upper bounds for x are respectively a and b then in the limit as n becomes infinite and the maximum Δx_i, $i = 1,2, \cdots, n$, approaches zero, the last summation on the right approaches zero, $f(x)$ being an ordinary frequency function. Thus in this limiting case

$$(32.11) = \int_a^b \int_a^b x_i^3 x_j^2 f(x_i) f(x_j) dx_i\, dx_j, \quad i \neq j,$$

$$= \int_a^b x_i^3 f(x_i) dx_i \cdot \int_a^b x_j^2 f(x_j)\, dx_j .$$

In general, under these limiting conditions, any summation $\sum_{i=1}^{n} x_i^k f^r(x_i) \overline{\Delta x_i}^r$ approaches zero, if r is greater than 1, k and r being positive integers. For let ε_n be the maximum Δx_i for specified n. Now $x_i^k f^r(x_i) \leqq M$ for $i = 1,2, \ldots, n$. Hence

$$\sum_{i=1}^{n} x_i^k f^r(x_i) \overline{\Delta x_i}^r \leqq M \sum_{i=1}^{n} \overline{\Delta x_i}^r \leqq M \varepsilon_n^{r-1} \sum_{i=1}^{n} \Delta x_i = M \varepsilon_n^{r-1} (b-a)$$

which approaches zero with ε_n. This establishes the well known statement that, the values of x being independent,

$$\int_a^b \int_a^b \cdots \int_a^b x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} f(x_{i_1}) f(x_{i_2}) \cdots f(x_{i_k})\, dx_{i_1} dx_{i_2} \cdots dx_{i_k}$$

$$= \int_a^b x_{i_1}^{a_1} f(x_{i_1}) dx_{i_1} \cdot \int_a^b x_{i_2}^{a_2} f(x_{i_2}) dx_{i_2} \cdots \int_a^b x_{i_k}^{a_k} f(x_{i_k}) dx_{i_k};$$

For, under the above limiting conditions, all those terms which contain a sum $s_{hk} = \sum x_i^h f^k(x_i) \overline{\Delta x_i}^k$ with k greater than 1 must vanish. By the last property of the D operators given in (2) it is seen that there is always only one term which does not contain such an s_{hk}; and from this term arises the product of the definite integrals.

A. R. Stoale

A GENERALIZED ERROR FUNCTION*

By

ALBERT WERTHEIMER

I. INTRODUCTION

Given a set of observed values l_i ($i = 1, 2, 3, \ldots, n$.) obtained from n observations assumed to be made on the same quantity, l, under the same conditions. We seek to determine two functions $f(P, l_i)$ and $\varphi(P, l_i)$ such that

$$f(P, l_i) = 0, \qquad (i = 1, 2, 3, \ldots, n)$$

defines p as a unique value assigned to the observed quantity; and $\varphi(P, l_i)dl_i$ gives to within infinitesimals of higher order the probability that if another observation is made, the observed value will lie in the interval

$$l_i \leqq l \leqq l_i + dl_i.$$

Gauss determined the φ function to be the so-called normal error law namely,

$$\varphi(P, l_i) = ce^{-h^2(P-l_i)^2}$$

on the basis of the following assumptions.

(a) The product $\prod_i \varphi(P, l_i)$ is to be a maximum with respect to p. Thus

$$\sum_i \frac{\partial}{\partial P} \log \varphi(P, l_i) = 0,$$

$$\sum_i \frac{\partial^2}{\partial P^2} \log \varphi(P, l_i) \neq 0.$$

*Presented to the American Mathematical Society, December 28, 1931.

(b) The unique value ρ is the arithmetic mean of the observations. Thus

$$f(P, l_i) = \sum_i (P - l_i).$$

(c) The probability function is a function of $(P - l_i)$. Thus

$$\phi(P, l_i) \equiv \phi(P - l_i).$$

Poincaré[1] on the basis of the first two assumptions only obtained the error function

$$\phi(P, l_i) = \theta(l_i) e^{W(P) + l_i V(P)},$$

where

$$\frac{dW}{dP} + P \frac{dV}{dP} = 0.$$

In this paper we assume the unique value ρ to be defined by a function satisfying certain conditions, and obtain on the basis of assumption (a) a more general error function from which the so-called normal error law, the Poincare function, and other forms of the error function as well as the Pearson curves are obtained as special cases.

2. *The unique value* ρ.

We now make the following assumptions:

I: The unique value ρ is defined explicitly as a function of the observed values in the region $a \leq l_i \leq b$. Thus

$$P - F(l_1, l_2, l_3, \cdots, l_n) = 0,$$

where F is single valued, continuous with continuous derivatives up to the second order.

II: The value of ρ is independent of the order in which the observations are obtained. Thus F is a symmetric function.

III: The change in ρ due to a change in one of the observed values, say l_i, is a function of ρ and l_i only. Thus

$$F_{l_i} = F_{l_i}(F, l_i).$$

[1] H. Poincare, *Calcul des Probabilites* (1912), p. 171.

IV: If p is regarded as a function of a single variable, say l_i, while all the others are regarded as constants, then with respect to this variable p is a monotonic function and is not constant in any portion of the interval in which it is defined. Thus

$$F_{l_i} \neq 0$$

for all i's.

We have then for the determination of the ϕ function the two equations

$$(1) \qquad \sum_i \frac{\partial}{\partial P} \log \phi(P, l_i) = 0,$$

$$(2) \qquad P - F(l_1, l_2, l_3, \cdots, l_n) = 0,$$

which must be simultaneously satisfied for any set of values in the region defined.

3. *The g function.*

We will now show by means of the following theorems that if F satisfies the given conditions, then there exists a unique function $g(P, l_i)$ such that the equation

$$\sum_i g(P, l_i) = 0,$$

is identical with equation (2).

THEOREM I.

Given a function of n variables,

$$F(x', x^2, x^3, \cdots, x^n),$$

continuous with continuous non-vanishing first derivatives in the

region defined, such that

$$F_{x^i} = \psi^i(F, x^i);$$

then $\psi^i(F, x^i)$ must be in the form of a product of a function of F and a function of t_i. Thus

$$\psi^i(F, x^i) = \omega(F)\beta^i(x^i).$$

Proof:
We have

$$F_{x^i x^j} = \psi_F^i(F, x^i)\psi^j(F, x^j),$$

and

$$F_{x^j x^i} = \psi_F^j(F, x^i)\psi^i(F, x^i).$$

Hence:

$$\frac{\psi_F^i(F, x^i)}{\psi^i(F, x^i)} = \frac{\psi_F^j(F, x^j)}{\psi^j(F, x^j)} = \cdots = \frac{\psi_F^n(F, x^n)}{\psi^n(F, x^n)} = \eta(F).$$

Integrating, we get

$$\log\psi^i(F, x^i) = \int \eta(F)dF + \xi^i(x^i).$$

from which it follows that

$$\psi^i(F, x^i) = \omega(F)\beta^i(x^i).$$

THEOREM II.

Given a function of n variables,

$$F(x^1, x^2, x^3, \ldots, x^n),$$

continuous with continuous non-vanishing first derivatives in the region defined, then in order that there shall exist a unique function $\mathcal{E}(F)$ such that

$$\xi(F) \equiv \sum_i u^i(x^i),$$

it is necessary and sufficient that

$$\frac{F_{x^i}}{F_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Proof:

Necessary conditions:—

If the ξ function exists then the functional matrix

$$\begin{Vmatrix} u'_{x'} & u^2_{x^2} & u^3_{x^3} & \cdots & u^n_{x^n} \\ F_{x'} & F_{x^2} & F_{x^3} & \cdots & F_{x^n} \end{Vmatrix}$$

must be of rank one. Hence

$$\frac{F_{x^i}}{F_{x^j}} = \frac{u^i_{x^i}}{u^j_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Sufficient conditions:—

We assume that

$$\frac{F_{x^i}}{F_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Then we have the following identities:

a) $$\frac{\partial^2}{\partial_{x^i}\,\partial_{x^j}} \log \frac{F_i}{F_j} = 0,$$

b) $$\frac{F_i}{F_j} = \frac{F_{ik}}{F_{jk}} = \frac{F_{ik\ell}}{F_{jk\ell}},$$

for k, $\neq i$ or j and

c) $$F_{ij}\{F_{ki}\,F_i - F_{ii}\,F_k\} = F_i\{F_{ijk}\,F_i - F_{iij}\,F_k\},$$

where for convenience of notation,

$$F_i \equiv F_{x^i},\quad F_{ik} = F_{x^i x^k},\qquad \text{etc.}$$

Making use of a), b), and c), it is easily shown that the functional matrix

$$\begin{Vmatrix} \frac{\partial}{\partial x^1}\left(\frac{F_{ij}}{F_i F_j}\right) & \frac{\partial}{\partial x^2}\left(\frac{F_{ij}}{F_i F_j}\right) \cdots & \frac{\partial}{\partial x^n}\left(\frac{F_{ij}}{F_i F_j}\right) \\ F_1 & F_2 \quad \cdots & F_n \end{Vmatrix}$$

is of rank one. It follows that

d) $$\frac{F_{ij}}{F_i F_j} = \lambda(F).$$

Now the differential equation that defines the ξ function is

$$\xi_{ij} \equiv \xi_{FF} F_i F_j + \xi_F F_{ij} = 0,$$

or

$$\frac{\xi_{FF}}{\xi_F} = -\frac{F_{ij}}{F_i F_j},$$
$$= -\lambda(F) \text{ from d).}$$

Hence $\xi(F)$ is uniquely determined, namely,

$$\xi(F) = K\int e^{-\int \lambda(F) dF} dF + H.$$

where K and H are constants of integration.

Now, for our problem, if F satisfies the given conditions, we can apply the two theorems in succession and we have that there exists a unique function

$$\xi(F) \equiv u^i(l_i).$$

But due to the symmetry of F all the u^i functions will be the same and we have

$$\xi(F) \equiv u(l_i).$$

If we now define

$$g(\rho, l_i) \equiv \frac{1}{n}\xi(\rho) - u(l_i),$$

we have

$$\Sigma_i\, g(p, \ell_i) = \xi(p) - \xi(F) = 0.$$

4. *General Error Function*

We may now write equations (1) and (2) in the form respectively

$$\Sigma_i \frac{\partial}{\partial p} \log \varphi(p, \ell_i) = 0,$$

$$\Sigma_i\, g(p, \ell_i) = 0.$$

These equations must be simultaneously satisfied for an arbitrary set of values ℓ_i in the region defined. It follows that they are identical. Thus

$$\frac{\partial}{\partial p} \log \varphi(p, \ell_i) = \psi(p) g(p, \ell_i),$$

where $\psi(p)$ is an arbitrary function.
Integrating, we get

$$(3) \qquad \varphi(p, \ell_i) = \theta(\ell_i)\, e^{\int \psi(p) g(p, \ell_i)\, dp}$$

where $\theta(\ell_i)$ is an arbitrary function. This is our general error function. In order to insure a maximum we must have

$$(4) \qquad \psi(p) g_p \neq 0.$$

5. *A Generalized Normal Function*

If we now make the additional assumption that

$$\varphi(p, \ell_i) \equiv \varphi\{g(p, \ell_i)\},$$

we have

$$\begin{vmatrix} \varphi_p & \varphi_{\ell_i} \\ g_p & g_{\ell_i} \end{vmatrix} = 0$$

Expanding and simplifying, we get

$$\frac{\theta_{l_i}}{\theta g_{l_i}} = \frac{\psi(p) g(p, l_i)}{g_p} - \int \psi(p) dp.$$

Differentiating with respect to l_i, we get

$$\frac{1}{g_{l_i}} \frac{\partial}{\partial l_i} \left(\frac{\theta_{l_i}}{\theta g_{l_i}} \right) = \frac{\psi(p)}{g_p} = K.$$

Integrating and substituting in (3), we get

$$\phi(g) = c e^{k g^2}.$$

From (4) we have

$$K g_p^2 \neq 0,$$

Hence

$$\phi(g) = c e^{-h^2 g^2}. \tag{5}$$

We shall refer to this function as the "Generalized Normal Error Function".

5. *Application to Special Cases*

If p is defined as the arithmetic mean, then the region considered is $-\infty < l_i < +\infty$, and

$$g(p, l_i) = p - l_i.$$

The normal law is obtained directly from (5), and from (4) we have

$$\phi(p, l_i) = \theta(l_i) e^{\int \psi(p)(p - l_i) dp}$$
$$= \theta(l_i) e^{w(p) + l_i v(p)},$$

where

$$\frac{dW}{dp} + p\,\frac{dV}{dp} = 0,$$

which is the same as the Poincaré function.

For the geometric mean, the region considered is

$$0 < l_i < \infty$$

and

$$g(p, l_i) = \log p - \log l_i.$$

Hence, from (3)

$$\Phi(p, l_i) = \theta(l_i)\, e^{\int \psi(p)\{\log p - \log l_i\}\, dp},$$

and from (4)

$$\Phi\{\log p - \log l_i\} = c e^{-h^2\{\log p - \log l_i\}^2}.$$

The Geometric mean as the most probable value, as well as its generalized normal curve are used for certain astronomical photometric measurements.[1]

For the harmonic mean, the region considered is

$$(0 < l_i < \infty)$$

and

$$g(p, l_i) = \left(\frac{1}{p} - \frac{1}{l_i}\right).$$

Then from (3), we have

$$\Phi(p, l_i) = \theta(l_i)\, e^{\int \psi(p)\left\{\frac{1}{p} - \frac{1}{l_i}\right\} dp}$$

and from (4), we have

$$\Phi\left\{\frac{1}{p} - \frac{1}{l_i}\right\} = c e^{-h^2\left\{\frac{1}{p} - \frac{1}{l_i}\right\}^2}.$$

7. *Remarks About the Generalized Normal Curves*

Let us consider briefly some characteristics of the generalized normal curves corresponding to the following three special cases.

[1] Whittaker & Robinson, Calculus of observations (1924), p. 218.

(a) *Arithmetic mean:* Here

$$\phi(p,l) = ce^{-h^2(p-l)^2}$$

From this equation we see that

$$\phi(p,p+\epsilon^2) = \phi(p,p-\epsilon^2),$$
$$\phi(p,l) = \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = ce^{-h^2p^2},$$
$$\phi(p,\infty) = 0.$$

(b) *Harmonic mean:* In this case

$$\phi(p,l) = ce^{-h^2\{\frac{1}{p}-\frac{1}{l_i}\}^2},$$

from which we see that

$$\phi(p,p-\epsilon^2) < \phi(p,p+\epsilon^2)$$
$$\phi(p,l) < \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = 0,$$
$$\phi(p,\infty) = ce^{-\frac{h^2}{p^2}}$$

(c) *Geometric Mean:* Here

$$\phi(p,l) = ce^{-h^2\{\log p - \log l_i\}^2}$$

and

$$\phi(p,p-\epsilon^2) < \phi(p,p+\epsilon^2).$$
$$\phi(p,l) < \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = 0,$$
$$\phi(p,\infty) = 0,$$

Instead of treating these normal functions as three distinct error laws referred to the same measuring scale, we can regard them as a single error law with reference to three different measur-

ing scales (see sketch). This viewpoint helps to explain the above mentioned characteristics of these laws.

The law for the arithmetic mean applies when an object is measured with a uniformly graduated scale in ρ. The characteristics for this law follow directly from the consideration that the scale is everywhere the same.

The law for the harmonic mean holds when an object is measured with a reciprocally graduated scale, as for instance measuring the volume of a gas with a pressure gauge graduated for volume. In this case the scale becomes crowded as ρ increases, and hence

$$\varphi(\rho+\epsilon^2,\ t+\epsilon^2) > \varphi(\rho, t),$$

and also

$$\varphi(\rho, \rho+\epsilon^2) > \varphi(\rho, \rho-\epsilon^2).$$

For large values of ρ it would take only a small error in the reading of the scale to make an infinitely large error in the value of ρ and hence $\varphi(\rho, \infty)$ does not necessarily vanish. On the other hand the zero point is at an infinite distance and hence $\varphi(\rho, 0)=0$.

The law for the geometric mean holds for measuring objects with a logarithmically graduated scale. The same remarks as for the harmonic mean apply here, except that in this case it would take an infinitely large error in the reading of the scale to make an infinitely large error in the value of ρ. Hence $\varphi(\rho, \infty)=0$.

8. *The Pearson Curves*

Leaving out the subscripts in (3), we have for the general error function

$$\varphi(\rho, t) = \theta(t)\, e^{\int \psi(\rho) g(\rho, t)\, d\rho}$$

Remembering that

$$g(p,\ell) \equiv \frac{1}{n}\,\xi(p) - u(\ell),$$

we have

$$\frac{\partial \varphi}{\partial \ell} = \varphi\left\{\frac{\varphi_\ell}{\theta} + u(\ell)\int \psi(p)\,dp\right\}.$$

Thus for a given p the curve approaches the ℓ axis asymptotically.

Let us now impose the condition that

$$\left.\frac{\partial \varphi}{\partial \ell}\right|_{\ell=p} = 0,$$

then

$$\frac{\theta_\ell}{\theta} = u(\ell)\int \psi(\ell)\,d\ell.$$

Integrating, we get

$$\theta(\ell) = ce^{\int u(\ell)\{\int \psi(\ell)\,d\ell\}\,d\ell},$$

so that

$$(6)\quad \varphi(p,\ell) = ce^{-\int u(\ell)\{\int \psi(\ell)\,d\ell\}\,d\ell + \int \psi(p) g(p,\ell)\,dp}$$

and

$$\frac{\partial \varphi}{\partial \ell} = \varphi\, u_\ell \int_\ell^p \psi(t)\,dt,$$

where t is a variable of integration. If we now take as a special case

$$\psi(t) = 1,$$
$$u_\ell = \{b_0 + b_1\ell + b_2\ell^2 + \cdots\}^{-1},$$

we have

$$\frac{\partial \varphi}{\partial \ell} = \varphi \frac{\rho - \ell}{b_0 + b_1 \ell + b_2 \ell^2},$$

which is the differential equation defining the Pearson system of frequency curves. For this case, (6) reduces to

$$\varphi(\rho, \ell) = c e^{\int \frac{\rho - \ell}{b_0 + b_1 \ell + b_2 \ell^2} d\ell + \frac{1}{n} \int \xi(\rho) d\rho},$$

from which we see that by a proper choice of $\xi(\rho)$ we can choose the value of ρ for which the product $\prod_i \varphi(\rho, \ell_i)$ shall be a maximum.

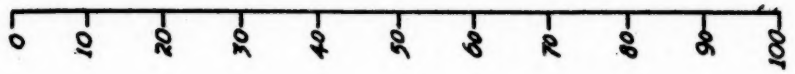
It may be noted that the differential equation defining the Pearson curves is often derived on the basis of the assumptions that the curve shall approach the ℓ axis asymptotically, and have only one maximum point.

In conclusion, it appears that if we restrict the function that defines to satisfy the assumptions given in this paper, and also impose the condition that ρ shall be the most probable value in the sense, that the product

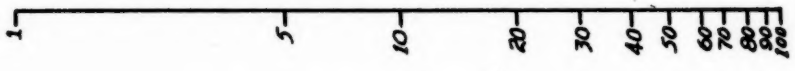
$$\prod_i \varphi(\rho, \ell_i)$$

shall be a maximum with respect to ρ, then (3) is the most general form of the error function.

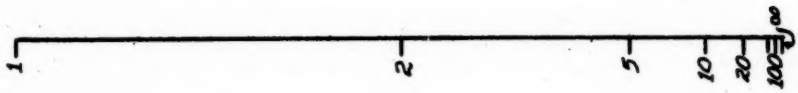
Scales, corresponding to the arithmetic, geometric, and harmonic means.



Arithmetic Mean — Uniform Scale



Geometric Mean — Logarithmic Scale



Harmonic Mean — Reciprocal Scale

albert wertheimer

Errata (Annals Vol. III, No. 1)

Page 46, Equation (3) should read

$$y = \frac{N}{\sigma\sqrt{2\pi}} f'(x) e^{\frac{-[f(x)-M]^2}{2\sigma^2}}$$

instead of

$$y \frac{N}{\sigma\sqrt{2\pi}} f'(x) e^{\frac{-[f(x)-M]^2}{2\sigma^2}}$$

Page 50, Table I. For the first step-interval the constant K should be negative, that is, $-.30102\ 99956$
For step-interval 9, K should be .65321 25137 instead of .64321 25137.

Page 51. the equation should read

$$\sigma_g = \sqrt{\frac{\Sigma F \log^2 x}{N} - C^2}$$

instead of

$$\sigma_g = \sqrt{\frac{\Sigma F \log^2 x}{N} - C^2}$$

Page 52, first line, should read $F \log x$'s instead of $F \log^2 x$'s

Page 55, eighth line, should read "of the number and r is the remainder"
Instead of "of the number and is the remainder."